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A WYLER-TYPE APPROACH TO CATEGORICAL TOPOLOGY

By

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INTRODUCTION

The concept of a topological category, that is, a concrete category which admits initial structures, plays a basic role in Categorical Topology. In the early 1970's, O. Wyler ([Wyler 1971a, b]) interpreted every topological category $(\mathscr{A}, U)$ over a base category $\mathscr{X}$ as a category $\mathrm{Mod}(T)$ of T–models with respect to a theory T, i.e. a functor from $\mathscr{X}$ into the category of complete lattices. In other words, Wyler established a model–theoretic correspondence between such theories and topological categories.

Several natural questions arise concerning this model–theoretic correspondence. Firstly, starting from the theoretic side, it is of interest to consider the categories $\mathrm{Mod}(T)$ obtained by varying the codomain categories of the functors T. Specifically, categories of T–models corresponding to theories into certain categories of ordered sets may be investigated. In addition, one may ask for a theoretic interpretation of certain topological completions, e.g., for the Mac Neille completion. Finally, and this is the main objective of the thesis, one may ask for an extension of Wyler's model–theoretic correspondence to important types of topological categories (e.g. cartesian closed topological categories, universally topological categories). The given theoretic descriptions of these categories exhibit a common form, namely, in each case, the corresponding theory sends pullbacks into diagrams of a special type, emphasising also in theoretic terms the basic role of pullbacks in Categorical Topology.

We should mention that, in this thesis, we follow a one–step constructive approach, which complements the two–step procedure of characterising topological categories within functor–structured categories, as carried out in the fundamental paper, *Universal Topology*, by H. Herrlich (cf. [Herrlich 1984]), which in fact inspired the main ideas of this thesis.

We give a survey of the subsequent Chapters :

Chapter 0 contains a summary of well–known terminology which will subsequently be used in the thesis.

In **Chapter 1** we begin by describing how topological categories may be viewed as categories of models corresponding to theories into the category of complete lattices. This leads naturally to the study of categories of T–models corresponding to theories into categories other than the category of complete lattices. It is shown, for example, that a concrete category corresponds to a poset–valued theory just in the case that it is (co)fibration complete. This shows that a concrete category is of the form $\mathrm{Mod}(T)$ only for poset–valued theories T .

We make some technical observations regarding the correspondence between transformations and concrete functors. In particular, the fact that natural transformations between theories are in a bijective correspondence to finality preserving concrete functors between their respective categories of models will be of importance in Chapter 2. A theoretic interpretation is given of those categories which are (co)reflective modifications of certain concrete categories.

Chapter 2 deals with the theoretic interpretation of certain topological completions of concrete categories. These are described in abstract theoretic terms using the correspondence between transformations and concrete functors. We also consider how concrete categories are embedded into (co)fibration complete categories. These "weak" completions have the nice property that they are always legitimate. For an arbitrary concrete category, the relationship between its topological completions and the various order-theoretic completions of its fibres is rather weak. However, if one assumes some additional structure properties, such as (co)fibration completeness, then the concepts of a categorical completion and an order-theoretic completion are more closely related, as shown by the result that for certain kinds of cofibrations, taking the universal order-theoretic completion of each fibre even yields the universal final topological completion.

Chapter 3 is entirely concerned with the main goal of this thesis. We study so-called "convenient" topological categories, i.e., topological categories with additional structure. The purpose is to characterise each such type of category as a category of T-models for some theory T which satisfies a special "preservation" property with respect to pullbacks. The cartesian closed topological categories are characterised as those categories of T-models where the associated theory T sends a pointwise pullback of any regular sink into product covering family of diagrams. The concretely cartesian closed topological categories are characterised as those for which the associated theory T sends the pointwise pullback of an arbitrary sink into a product covering family. We also characterise the concretely cartesian closed categories by means of a certain natural transformation, given by the product of two structures.

Perhaps the most satisfactory result of this Chapter is the characterisation of the universally topological categories. The theories corresponding to these categories may be described in two ways : firstly, they are shown to be frame-valued, send pullbacks into covering diagrams, and send morphisms into downset-preserving, cover-reflecting maps; secondly, they are shown to send the pointwise pullback of any sink into an order-covering diagram. Similarly, the concrete quasitopoi may be characterised by those theories which send the pullback of any regular sink into an order-covering family of diagrams.

Finally, we consider hereditary topological categories. These are characterised as categories of T–models for which the theory T preserves terminal objects and sends the pointwise pullback of an arbitrary sink along an embedding into a weakly covering diagram family. In this context, a notion of strong heredity is introduced and characterised by a frame–valued theory sending pullbacks along monomorphisms into order–covering diagrams.

CHAPTER 0

PRELIMINARIES

This chapter contains a summary of well-known categorical concepts and results which will be needed in subsequent chapters. We restrict our attention to terminology occurring in the area of categorical topology. For more general categorical notions, the reader is referred to [Herrlich and Strecker 1973], or [Mac Lane 1971].

We use the set-theoretic terminology and assumptions in [Herrlich and Strecker 1973, pp. 328 – 331], that is, we use a hierarchy consisting of sets, classes and conglomerates, where every set is a class and every class a conglomerate. A conglomerate C is called *legitimate* (resp. *small*) if there exists an injection from C into a class (resp. set).

Concrete Categories

Let $\mathcal{X}$ be a category. A *concrete category* over $\mathcal{X}$ is a pair $(\mathcal{A}, U)$ where $\mathcal{A}$ is a category and $U: \mathcal{A} \longrightarrow \mathcal{X}$ is a faithful and amnestic functor (a faithful functor U is called *amnestic* provided that any $\mathcal{A}$-isomorphism f is an $\mathcal{A}$-identity whenever $U(f)$ is an $\mathcal{X}$-identity). A *concrete functor* $F: (\mathcal{A}, U) \longrightarrow (\mathcal{B}, V)$ between concrete categories over $\mathcal{X}$ is a functor such that $U = V \cdot F$.

For a concrete category $(\mathcal{A}, U)$ over $\mathcal{X}$ and an $\mathcal{X}$-object X , the *fibre* of X , denoted $U^{-1}[X]$, is defined to be the class of all $\mathscr{A}$-objects A such that $UA = X$. If, for each X in $\mathscr{X}$, $U^{-1}[X]$ is a set, then $(\mathscr{A},U)$ is said to be *small-fibred* (or, *fibre-small*). For the purposes of this thesis, all concrete categories, unless otherwise stated, are assumed to be fibre-small. For each X, $U^{-1}[X]$ can be ordered by $A \leq B$ (A is *finer than* B) iff there is an $\mathscr{A}$-morphism $a : A \longrightarrow B$ such that $U(a) = id_X$. (In general, the relation $\leq$ is a preorder; the amnesticity of U ensures that $\leq$ is a partial ordering.)

Let $(\mathscr{A},U)$ be a concrete category over $\mathscr{X}$. A *U-morphism* is a pair (f,A), where $f : UA \longrightarrow X$ is an $\mathscr{X}$-morphism and A an $\mathscr{A}$-object. A *U-sink* on X is a pair (X,S), where X is an $\mathscr{X}$-object and $S = (f_i : UA_i \longrightarrow X)_{i \in I}$ is a family of U-morphisms indexed by some class I. A *U-source* $(f_i : X \longrightarrow UA_i)_{i \in I}$ is dually defined. $(\mathscr{A},U)$ is said to be *finally complete* provided any U-sink $(f_i : UA_i \longrightarrow X)_{i \in I}$ has a unique *U-final* lift $(a_i : A_i \longrightarrow A)_I$, i.e., for any $\mathscr{A}$-sink $(b_i : A_i \longrightarrow B)_I$ and any morphism $g : UA \longrightarrow UB$ with $g \cdot U(a_i) = U(b_i)$ for each i, there exists a unique $\mathscr{A}$-morphism $h : A \longrightarrow B$ such that $U(h) = g$ and $h \cdot a_i = b_i$ for each i. (Dual notions : *initially complete*, *initial lift*.) It is well-known that any concrete category is finally complete iff it is initially complete.

A concrete category $(\mathscr{A},U)$ is called *lift-finally* (resp. *lift-initially*) *complete* iff every U-sink (resp. every U-source) which can be lifted to an $\mathscr{A}$-sink (resp. $\mathscr{A}$-source) has a final (resp. initial) lift.

A full concrete subcategory $(\mathscr{A},U)$ of a finally complete category $(\mathscr{B},V)$ is said to be *finally closed* in $(\mathscr{B},V)$ provided for any final $\mathscr{B}$-sink $(a_i : A_i \longrightarrow A)_I$ with each A_i an $\mathscr{A}$-object, it follows that A is an $\mathscr{A}$-object.

A concrete category $(\mathscr{A},U)$ over $\mathscr{X}$ is called *topological* provided :

(1) $(\mathscr{A},U)$ is finally complete, and

(2) $(\mathscr{A},U)$ is small–fibred.

Well–known examples of topological categories include **Top** (topological spaces), **Unif** (uniform spaces), **Prox** (proximity spaces) and **Lim** (limit spaces).

Completions Of Concrete Categories

The terminology in this section comes primarily from [Adámek, Herrlich, Strecker 1979a] and [Herrlich 1984].

A concrete functor $F:(\mathscr{A},U)\to(\mathscr{B},V)$ is said to be *finally dense* if each $\mathscr{B}$–object B is the V–final lift of any sink of the form $(f_i : VF(A_i)\longrightarrow X)_I$. F is said to be *finality preserving* if for each final $\mathscr{A}$–sink $(a_i : A_i\longrightarrow A)_I$, the sink $(F(a_i) : FA_i\longrightarrow FA)_I$ is final in $\mathscr{B}$. A *final completion* of a concrete category $(\mathscr{A},U)$ is a finally dense concrete full embedding $E:(\mathscr{A},U)\to(\mathscr{B},V)$, where $(\mathscr{B},V)$ is finally complete. (Dual notions : *initially dense* , *initiality preserving* , *initial completion* .)

Final completions $E_1:(\mathscr{A},U)\to(\mathscr{A}_1,U_1)$ and $E_2:(\mathscr{A},U)\to(\mathscr{A}_2,U_2)$ of $(\mathscr{A},U)$ can be ordered by the following equivalent conditions :

(1) $E_1\le E_2$ iff there exists a concrete full embedding $E:(\mathscr{A}_1,U_1)\to(\mathscr{A}_2,U_2)$ such that $E_2 = E\cdot E_1$,

(2) $E_1\le E_2$ iff there exists a finality preserving concrete functor $F:(\mathscr{A}_2,U_2)\to(\mathscr{A}_1,U_1)$ with $E_1 = F\cdot E_2$,

(3) $E_1\le E_2$ iff there exist concrete functors $E:(\mathscr{A}_1,U_1)\to(\mathscr{A}_2,U_2)$ and $F:(\mathscr{A}_2,U_2)\to(\mathscr{A}_1,U_1)$ with $E_2 = E\cdot E_1$, $E_1 = F\cdot E_2$ and $F\cdot E = Id_{\mathscr{A}_1}$.

The second of the above conditions will prove most suitable for the purposes of this thesis. Note that the relation $\leq$ is a preorder, hence by a "*smallest*" (resp. "*largest*") *final completion* is meant a final completion which is smaller (resp. larger) than any other completion and such that there is no properly smaller (resp. larger) final completion. Dual notions : "*smallest*" (resp. "*largest*") *initial completion* .

In general, a given concrete category $(\mathcal{A}, U)$ may fail to have a final completion. If $(\mathcal{A}, U)$ has a final completion, then it has a smallest final completion $E^*: (\mathcal{A}, U) \to (\mathcal{A}^*, U^*)$, called the *Mac Neille completion* of $(\mathcal{A}, U)$. (When the context is clear, we shall simply write $(\mathcal{A}^*, U^*)$ for the Mac Neille completion.) In case $(\mathcal{A}^*, U^*)$ exists, the embedding E^* has the property of being both initially and finally dense. It is well–known that $(\mathcal{A}^*, U^*)$ is simultaneously a smallest initial completion of $(\mathcal{A}, U)$.

Given a final completion $E: (\mathcal{A}, U) \to (\mathcal{B}, V)$ of $(\mathcal{A}, U)$, $(\mathcal{A}^*, U^*)$ may be obtained as the *initial hull* of $(\mathcal{A}, U)$ in $(\mathcal{B}, V)$, i.e., as the full subcategory of $(\mathcal{B}, V)$ containing those $\mathcal{B}$–objects which are initial lifts of sources of the form $(f_i : X \longrightarrow VE(A_i))_I$. (Dual notion : *final hull* .)

The *universal final completion* of $(\mathcal{A}, U)$, if it exists, is denoted by $\mathrm{Univ}(\mathcal{A})$ and is the largest finality preserving final completion of $(\mathcal{A}, U)$, i.e., the embedding $E: (\mathcal{A}, U) \to \mathrm{Univ}(\mathcal{A})$ preserves finality, and any finality preserving concrete functor $F: (\mathcal{A}, U) \to (\mathcal{B}, V)$ with $(\mathcal{B}, V)$ finally complete has a unique finality preserving concrete extension $G: \mathrm{Univ}(\mathcal{A}) \to (\mathcal{B}, V)$.

The *largest final completion*, if it exists, of a given concrete category $(\mathcal{A}, U)$, is denoted by $\mathrm{Siev}(\mathcal{A})$. The embedding $E: (\mathcal{A}, U) \to \mathrm{Siev}(\mathcal{A})$ has the following universal property : for any concrete functor $F: (\mathcal{A}, U) \to (\mathcal{B}, V)$ with $(\mathcal{B}, V)$ finally complete, there exists a unique finality preserving concrete functor $\bar{F} : \mathrm{Siev}(\mathscr{A}) \to (\mathscr{B}, V)$ such that $F = \bar{F} \cdot E$.

Mac Neille completions of small concrete categories may be constructed as categories of *closed sinks* . For categories which are not small, this construction may yield a *quasicategory* (i.e., a "category" of which the collection of objects cannot be coded by a class). Given $(\mathscr{A}, U)$, for each U–sink $S = (f_i : UA_i \longrightarrow X)_I$, let S^{op} denote the U–source of all $g : X \longrightarrow UB$ such that for each $i \in I$, $g \cdot f_i : A_i \longrightarrow B$ is an $\mathscr{A}$–morphism. Analagously, for each U–source S , the opposite sink S^{op} may be obtained. A U–sink S is called *closed* iff $(S^{\mathrm{op}})^{\mathrm{op}} = S$. For a given sink S , $(S^{\mathrm{op}})^{\mathrm{op}}$ is the smallest closed sink containing S , and is called the *closed hull* of S . Given U–sinks $S = (f_i : UA_i \longrightarrow X)_I$ and $T = (g_j : UB_j \longrightarrow Y)_J$, a *sink map* is a map $p : X \longrightarrow Y$ such that for each $i \in I$ there exists a $j \in J$ with $p \cdot f_i : UA_i \longrightarrow Y = g_j : UB_j \longrightarrow Y$. If the conglomerate of closed sinks is legitimate, then we may consider a concrete category $\mathscr{L}$ which has the closed sinks as objects and the sink maps as morphisms. In fact, $\mathscr{L}$ is the Mac Neille completion $(\mathscr{A}^*, U^*)$ of $(\mathscr{A}, U)$ (cf. [Adámek, Herrlich, Strecker 1979a]); the assignment of an $\mathscr{A}$–object A to the sink of all U–morphisms $f : UB \longrightarrow UA$ such that f underlies some $\mathscr{A}$–morphism $h : B \longrightarrow A$, gives rise to a concrete full embedding of $(\mathscr{A}, U)$ into $\mathscr{L}$.

Similarly, the notion of a *semi–closed* (resp. *weakly–closed*) sink may be defined. Given $(\mathscr{A}, U)$, if the conglomerate of semi–closed (resp. weakly–closed) sinks is legitimate, then $\mathrm{Univ}(\mathscr{A})$ (resp. $\mathrm{Siev}(\mathscr{A})$) is obtained as the category of semi–closed (resp. weakly–closed) sinks and sink maps. For further details, see [Adámek, Herrlich, Strecker 1979a].

A concrete category $(\mathscr{A}, U)$ over $\mathscr{X}$ is called *strongly co–fibre–small* iff for each $X \in Ob(\mathscr{X})$ the conglomerate of all closed sinks into X is small; $(\mathscr{A}, U)$ is called *very* *strongly co–fibre–small* (resp. *extremely strongly co–fibre–small*) iff for each $X \in Ob(\mathcal{X})$ the conglomerate of all semi–closed sinks (resp. weakly–closed sinks) into X is small.

Convenient Categories

The terminology in this section is taken from [Adámek, Herrlich 1985].

An abstract category $\mathcal{X}$ is called *cartesian closed* if it has finite products and for each $\mathcal{X}$–object X the functor $X \times -$ has a right adjoint. The values of the right adjoint are called *power objects*, and are denoted by Y^X. The associated couniversal morphism $ev : X \times Y^X \longrightarrow Y$ is called the *evaluation*.

An $\mathcal{X}$–epimorphism $f : X \longrightarrow Y$ is called *regular* if f is the coequalizer of two $\mathcal{X}$–morphisms. A sink $(f_i : X_i \longrightarrow X)_I$ in $\mathcal{X}$ is called *regular* if there exists a subset $J \subset I$ such that the canonical morphism $\coprod_J f_j : \coprod_J X_j \longrightarrow X$ is a regular epimorphism. A sink $(f_i : X_i \longrightarrow X)_I$ in **Set** is regular iff it is an epi–sink, i.e., $X = \bigcup_I f_i[X_i]$. A category $\mathcal{X}$ is said to have *regular sink factorisations* if $\mathcal{X}$ is cocomplete and for each sink $(f_i : X_i \longrightarrow X)_I$ there exists a monomorphism $m : Y \longrightarrow X$ and a regular sink $(e_i : X_i \longrightarrow Y)_I$ with $f_i = m \cdot e_i$ for each $i \in I$.

If $\mathcal{X}$ has finite products, then regular sinks are said to be *finitely productive* provided that for each regular sink $(f_i : X_i \longrightarrow X)_I$ and each $\mathcal{X}$–object Y, the sink $(f_i \times id_Y : X_i \times Y \longrightarrow X \times Y)_I$ is regular. The finite productivity of final sinks, colimits and regular epimorphisms is defined analagously.

If $(\mathcal{A}, U)$ is topological over $\mathcal{X}$ and $(\mathcal{A}, U)$ is cartesian closed, then $\mathcal{X}$ is cartesian closed. A cartesian closed topological category $(\mathscr{A}, U)$ is said to have *concrete powers* if U preserves the cartesian structure, i.e., for A , $B \in Ob(\mathscr{A})$, $U(B^A) = UB^{UA}$ and $U(ev : A \times B^A \longrightarrow B) = ev : UA \times UB^{UA} \longrightarrow UB$. A cartesian closed topological category with concrete powers is called *concretely cartesian closed* .

A monomorphism m in a category $\mathscr{X}$ is called a *strong monomorphism* provided that each commutative square

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "g"'] & W \arrow[d, "f"] \arrow[dl, dashed, "d"'] \\ Y \arrow[r, "m"'] & Z \end{tikzcd}$$

with e an epimorphism has a diagonal d , i.e. $g = d \cdot e$ and $f = m \cdot d$.

A *quasitopos* is a category $\mathscr{X}$ satisfying the following conditions :

(1) $\mathscr{X}$ has finite limits and colimits;

(2) $\mathscr{X}$ is cartesian closed;

(3) for each $\mathscr{X}$–object X there exists a strong monomorphism $t_X : X \longrightarrow X^*$ such that given a pair consisting of a strong monomorphism $m : Y \longrightarrow Z$ and a morphism $f : Y \longrightarrow X$, there exists a unique pullback in $\mathscr{X}$ of the following form :

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "m"'] & X \arrow[d, "t_X"] \\ Z \arrow[r, dashed] & X^* \end{tikzcd}$$

A concrete category $(\mathcal{A}, U)$ over Set is called a *c–category* (alternatively, a *construct*) if all constant maps between $\mathcal{A}$–objects are $\mathcal{A}$–morphisms, where a *constant map* is a function which factors through a set of cardinality one. A topological category $(\mathcal{A}, U)$ is a c–category iff the $\mathcal{A}$–fibre of every one–element set is trivial (i.e. consists of precisely one element).

CHAPTER 1

CONCRETE CATEGORIES DETERMINED BY THEORIES

1. Topological categories

For further details concerning the material in this section, the reader may consult [Wyler 1971a, b].

It was Wyler's discovery, at the beginning of Categorical Topology, that the contravariant (fibre–)functor T from **Set** into the category **ClLatt** of complete lattices and infima–preserving maps, which assigns to every set X the set of all topologies on X, contains all the essential information about **Top**, the category of topological spaces and continuous maps. Wyler was able to associate with any functor $T : \mathscr{X}^{\mathrm{op}} \to$ **ClLatt** (called a *topological theory*) a category $\mathrm{Mod}(T)$ of T–models which, in the case of the fibre–functor $T : X \mapsto \{ \tau \mid \tau$ is a topology on $X \}$, coincides with **Top**. More generally, he showed that there is a model–theoretic correspondence between topological theories $\mathscr{X} \to$ **ClLatt** and topological categories over $\mathscr{X}$.

1.1 Topological categories are usually defined axiomatically by means of categorical concepts, as has been outlined in the previous chapter. However, it is also well–known that every topological category has the following property : each fibre (i.e. class of structures on a given underlying object) is a *complete lattice*. As our basis example, we consider the category **Top** over **Set** : the set of all topologies on a set X (which we denote by TX) is a complete lattice with respect to the *finer than* relation $\leq$ (given by $\tau \leq \bar{\tau}$ iff $\bar{\tau} \subset \tau$);

the largest (resp. smallest) element of TX is the indiscrete (resp. discrete) topology. A map $g : (X,\tau) \longrightarrow (Y,\sigma)$ ($\tau \in TX$, $\sigma \in TY$) is continuous iff the final topology with respect to $g : (X,\tau) \longrightarrow Y$ (denoted by $Tg(\tau)$) is finer than σ . It can be verified that the assignment $\tau \mapsto Tg(\tau)$ defines a morphism $Tg : TX \longrightarrow TY$ of partially-ordered sets which in addition preserves arbitrary suprema, but not in general infima. Finally, it is routine to verify that the assignments $X \mapsto TX$ (for a set X), $g \mapsto Tg$ (for a map g) define a functor T from **Set** to **CSLatt** – the category of complete lattices and suprema-preserving morphisms. These basic observations led Wyler to the following :

1.2 Definition ([Wyler 1971a, b])

(1) A *topological theory* on a category $\mathscr{X}$ is a functor $T : \mathscr{X} \to$ **CSLatt** . For each object X of $\mathscr{X}$, TX is a complete lattice, of which the underlying order $\leq$ is referred to as the *finer than* relation. For each $\mathscr{X}$-morphism $f : X \longrightarrow Y$, $Tf : TX \longrightarrow TY$ preserves arbitrary suprema.

(2) Every topological theory T on a category $\mathscr{X}$ induces a concrete category over $\mathscr{X}$, called a *top category* (alternatively, the *category of T-models*), denoted by Mod(T), as follows :

(i) objects of Mod(T) , called *T-models*, are pairs (X,α) , where X is an $\mathscr{X}$-object and $\alpha \in TX$,

(ii) an $\mathscr{X}$-morphism $f : X \longrightarrow Y$ is a *T-morphism* $f : (X,\alpha) \longrightarrow (Y,\beta)$ provided $Tf(\alpha) \leq \beta$; composition in Mod(T) is lifted from $\mathscr{X}$. Observe, given $f : X \longrightarrow Y$ and $\alpha \in TX$ that $f : (X,\alpha) \longrightarrow (Y, Tf(\alpha))$ is final in Mod(T) .

(iii) the associated forgetful functor $U_T : \mathrm{Mod}(T) \to \mathscr{X}$ is defined by : $U_T((X,\alpha)) = X$ for a T-model (X,α) , and $U_T(f) = f$ for a T-morphism $f : (X,\alpha) \longrightarrow (Y,\beta)$.

1.3 Examples (1) For the terminal (i.e., the one-morphism) category $\odot$, the categories Mod(T) induced by theories $T : \odot \to$ **CSLatt** are the complete lattices.

(2) An analagous procedure to that used in 1.1 for **Top** can be applied to yield **Unif**, Prox, **Lim** and other standard topological categories as top categories corresponding to topological theories.

(3) Given a set X , let RX be the power set of $X \times X$, ordered by inclusion. R defines a functor **Set** $\to$ **CSLatt** as follows : for any map $f : X \longrightarrow Y$ and $\rho \in RX$, $Rf : RX \longrightarrow RY$ is given by the assignment $\rho \mapsto (f \times f)[\rho]$. The induced category Mod(R) is (up to concrete isomorphism) the category **Rel** of binary relations.

1.4 Remarks (1) Wyler's original definition of a top category is actually based on functors $T : \mathscr{X}^{\mathrm{op}} \to$ **CILatt** (which we call *dual theories*); the T–models with respect to a given dual theory T are pairs (X, α) with $\alpha \in TX$, and the T–morphisms $f : (X, \alpha) \longrightarrow (Y, \beta)$ are $\mathscr{X}$–morphisms $f : X \longrightarrow Y$ with $\alpha \leq Tf(\beta)$. In contrast to 1.2, $Tf(\beta)$ is the initial structure on X with respect to $f : X \longrightarrow (Y, \beta)$.

(2) Topological theories arise in dual pairs (cf. [Wyler 1971a, b]) : for a topological theory $T : \mathscr{X} \to$ **CSLatt** and an $\mathscr{X}$–morphism $f : X \longrightarrow Y$, let $\bar{T}f(\tau) = \bigvee \{ \sigma \in TX \mid Tf(\sigma) \leq \tau \}$, where $\tau \in TY$ (here $\bigvee$ denotes supremum). It can be shown that $\bar{T}f : TY \longrightarrow TX$ preserves order, and that $Tf(\sigma) \leq \tau$ iff $\sigma \leq \bar{T}f(\tau)$. Thus $\bar{T}f$ is right adjoint to Tf (i.e. $(Tf, \bar{T}f)$ is a covariant Galois correspondence), hence $\bar{T}f$ preserves infima. So, $\bar{T}$ defines a dual theory $\mathscr{X}^{\mathrm{op}} \to$ **CILatt** , and we call $\bar{T}$ *the dual theory of* T .

The next result shows that the general theory of top categories is self–dual :

Theorem ([Wyler 1971a]) Mod($\bar{T}$) *is a top category over* $\mathscr{X}$, *concretely isomorphic to the top category* Mod(T) . □

We have seen that certain topological categories can be constructed by considering appropriate topological theories, but the question arises whether a given topological category can be characterised as a category of T–models for some topological theory T. The positive answer is provided by the following theorem, a result which, in effect, constitutes the starting point of this thesis.

1.5 Theorem ([Grothendieck 1961], [Wyler 1971b], [Richter 1979]) *The following conditions are equivalent, for a concrete category* $(\mathcal{A}, U)$ *over a category* $\mathcal{X}$:

(1) $(\mathcal{A}, U)$ *is topological*

(2) $(\mathcal{A}, U)$ *is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some topological theory* $T : \mathcal{X} \to \mathbf{CSLatt}$.

Proof $(1) \Rightarrow (2)$: By definition, $(\mathcal{A}, U)$ has small fibres. For an $\mathcal{X}$–object X, define a partial ordering $\leq$ on $U^{-1}[X]$ as follows : $A \leq \bar{A}$ iff there is some $\mathcal{A}$–morphism $g : A \longrightarrow \bar{A}$ such that $U(g) = id_X$, where $A, \bar{A} \in U^{-1}[X]$. Reflexivity and transitivity of $\leq$ follow immediately, while antisymmetry follows from the amnesticity of $(\mathcal{A}, U)$. Moreover, $U^{-1}[X]$ is a complete lattice : for a family $\{A_i\}_I$ in $U^{-1}[X]$, $\bigvee_I A_i$ is given by the final lift of the sink $(id_X : UA_i \longrightarrow X)_I$. So, for an $\mathcal{X}$–object X , put $TX = (U^{-1}[X], \leq)$. For an $\mathcal{X}$–morphism $f : X \longrightarrow Y$, and an element $A \in U^{-1}[X]$, define $Tf : TX \longrightarrow TY$ by the assignment $A \mapsto$ the final lift of the structured morphism $f : UA \longrightarrow Y$. Tf preserves suprema : if $\{A_i\}_I$ is a family in $U^{-1}[X]$, then we obtain an identity–carried final sink $(g_i : A_i \longrightarrow (\bigvee A_i))_I$, since TX is complete, and it follows from the finality of $(g_i)_I$ that $Tf(\bigvee A_i) = \bigvee Tf(A_i)$. These assignments define a topological theory $T : \mathcal{X} \to \mathbf{CSLatt}$, and associated top category $\mathrm{Mod}(T)$ with objects pairs (X, A) , $X \in Ob(\mathcal{X})$, $A \in U^{-1}[X]$, and morphisms $f : (X, A) \longrightarrow (Y, B)$ those $\mathcal{X}$–morphisms $f : X \longrightarrow Y$ where $Tf(A) \leq B$. Now define the concrete functor $G : \mathcal{A} \to \mathrm{Mod}(T)$ as follows : put $G(A) = (UA, A)$ for $A \in Ob(\mathcal{A})$, and $G(a) = U(a)$ for $a : A \longrightarrow B$ in $Mor(\mathcal{A})$. Clearly $U_T \cdot G = U$, and it is straightforward to verify that $G : \mathcal{A} \to \mathrm{Mod}(T)$ is indeed a concrete isomorphism over $\mathcal{X}$.

(2) $\Rightarrow$ (1) : It is sufficient to show that $(\mathrm{Mod}(T), U_T)$ is topological over $\mathscr{X}$. Clearly $\mathrm{Mod}(T)$ is small-fibred. Now we must show that $\mathrm{Mod}(T)$ is finally complete : let $(f_i : (X_i, \tau_i) \longrightarrow X)_I$ be a structured sink in $\mathrm{Mod}(T)$. Set $\tau = \bigvee_I Tf_i(\tau_i)$. Now $(f_i : (X_i, \tau_i) \longrightarrow (X, \tau))_I$ is final, for if $(g_i : (X_i, \tau_i) \longrightarrow (Y, \sigma))_I$ is any $\mathrm{Mod}(T)$-sink, and $h : X \longrightarrow Y$ is an $\mathscr{X}$-morphism with $h \cdot f_i = g_i$ for each $i \in I$, then we have : $Th(\tau) = Th(\bigvee_I Tf_i(\tau_i)) = \bigvee_I Th(Tf_i(\tau_i)) = \bigvee_I T(h \cdot f_i)(\tau_i) = \bigvee_I Tg_i(\tau_i)$. So $Th(\tau) \leq \sigma$, since for each $i \in I$, $Tg_i(\tau_i) \leq \sigma$. □

1.6 Remarks (1) In addition, the above proof shows precisely how the theory induced by a given topological category $(\mathscr{A}, U)$ is constructed, namely, as a certain extension of the fibre association $X \mapsto (U^{-1}[X], \leq)$, $X \in Ob(\mathscr{X})$.

(2) Note that given a topological category of form $\mathrm{Mod}(T)$, for each $X \in Ob(\mathscr{X})$, $U_T^{-1}[X] = \{ (X, \tau) \mid \tau \in TX \}$, so, strictly speaking, $(id_X : (X, \tau_i) \longrightarrow (X, \tau))_I$ is final in $\mathrm{Mod}(T)$ iff $(X, \tau) = \bigvee_I (X, \tau_i)$ in $U_T^{-1}[X]$. However, since $U_T^{-1}[X]$ is isomorphic to TX, and $(X, \tau) = \bigvee_I (X, \tau_i)$ in $U_T^{-1}[X]$ iff $\tau = \bigvee_I \tau_i$ in TX, it is for our purposes more convenient to describe finality in $\mathrm{Mod}(T)$ in terms of TX, i.e., $(f_i : (X_i, \tau_i) \longrightarrow (X, \tau))_I$ is final in $\mathrm{Mod}(T)$ iff $\tau = \bigvee_I Tf_i(\tau_i)$ in TX.

(3) Given a structured source $(g_i : X \longrightarrow (X_i, \tau_i))_I$ in $\mathrm{Mod}(T)$, it can be verified that the initial structure on X with respect to $(g_i)_I$ is given by the structure $\tau = \bigvee \{ \mu \in TX \mid Tg_i(\mu) \leq \tau_i \text{ for all } i \in I \}$.

2. Poset-valued theories, fibrations and cofibrations

We consider the structure of those concrete categories over $\mathscr{X}$ induced by **Pos**-valued functors, where **Pos** denotes the category of partially-ordered sets and order-preserving maps. Our purpose is to place the topological theories and top categories of the previous section in a more general setting.

2.1 Definition (cf. 1.2)

(1) A *theory* on $\mathscr{X}$ is a functor $T : \mathscr{X} \to \mathbf{Pos}$.

(2) Every theory T on $\mathscr{X}$ induces a category Mod(T) of *T–models* and *T–morphisms,* defined as follows :

(i) objects are pairs (X, α) , where X is an $\mathscr{X}$–object and $\alpha \in TX$,

(ii) morphisms $f : (X, \alpha) \longrightarrow (Y, \beta)$ are those $\mathscr{X}$–morphisms $f : X \longrightarrow Y$ with $Tf(\alpha) \leq \beta$.

(3) Dually, one can obtain a category Mod(T) from a theory $T : \mathscr{X}^{\mathrm{op}} \to \mathbf{Pos}$, where a morphism $f : (X, \alpha) \longrightarrow (Y, \beta)$ is an $\mathscr{X}$–morphism $f : X \longrightarrow Y$ with $\alpha \leq Tf(\beta)$.

2.2 Remarks (1) Note that Pos–valued theories, unlike topological theories, need not occur in dual pairs (in the sense of 1.4 (2)) . However, we shall say that a theory $T : \mathscr{X} \to \mathbf{Pos}$ *has a dual theory* iff there exists a theory $\bar{T} : \mathscr{X}^{\mathrm{op}} \to \mathbf{Pos}$ such that for each X in $\mathscr{X}$, $TX = \bar{T}X$, and for each $\mathscr{X}$–morphism $f : X \longrightarrow Y$, $(Tf, \bar{T}f)$ is a Galois correspondence (see 1.4 (2)).

(2) Functors into subcategories of Pos other than **CSLatt** may also be considered. For example, Menu and Pultr [1974] have considered functors into **BCPos** – the category of bounded–complete posets (i.e. posets in which every subset that has an upper bound has a supremum) and suprema–preserving maps.

We shall characterise those concrete categories which are concretely isomorphic to Mod(T) , for Pos–valued theories T . For this purpose we use the concept of a fibration (dually, cofibration), which occured first in [Grothendieck 1961] (see also [Gray 1965]) and later in a more specific form in [Wyler 1971b]. Using the language of Chapter 0 :

2.3 Definition A concrete category $(\mathcal{A}, U)$ is called a *fibration* (resp. *cofibration*) iff every singleton U–source $f : X \longrightarrow UB$ (resp. singleton U–sink $g : UA \longrightarrow Y$) has an initial (resp. final) lift.

2.4 Proposition *For a concrete category $(\mathcal{A}, U)$ the following conditions are equivalent :*

(1) $(\mathcal{A}, U)$ is a cofibration (resp. fibration)

(2) $(\mathcal{A}, U)$ is concretely isomorphic to $\mathrm{Mod}(T)$ *for some theory $T : \mathcal{X} \to$* **Pos** *(resp. some theory $T : \mathcal{X}^{\mathrm{op}} \to$* **Pos** *).*

Proof We prove the cofibration version; the fibration version is analagous.

(1) ⇒ (2) : If $(\mathcal{A}, U)$ is a cofibration, then the functor described in 1.6 (1) defines a theory $T : \mathcal{X} \to \mathbf{Pos}$. The assignments $A \mapsto (UA, A)$, for an $\mathcal{A}$–object A , and $g \mapsto U(g)$ for an $\mathcal{A}$–morphism g , define a concrete isomorphism $H : \mathcal{A} \to \mathrm{Mod}(T)$.

(2) ⇒ (1) : It is sufficient to show that $\mathrm{Mod}(T)$ is a cofibration. Let $f : (X, \tau) \longrightarrow Y$ be a structured morphism. Then $f : (X, \tau) \longrightarrow (Y, Tf(\tau))$ is final : given $g : Y \longrightarrow (Z, \sigma)$ such that $g \cdot f : (X, \tau) \longrightarrow (Z, \sigma)$ is a T–morphism, we have $\mathrm{T}\,(g \cdot f)(\tau) \leq \sigma$, hence $Tg\,(Tf(\tau)) \leq \sigma$, i.e. $g : (Y, Tf(\tau)) \longrightarrow (Z, \sigma)$ is a T–morphism. □

2.5 Definition A *co–fibre–functor* of a concrete category $(\mathcal{A}, U)$ is a functor $T : \mathcal{X} \to \mathbf{Pos}$ with the following properties :

(i) for each $\mathcal{X}$–object X , $TX = (U^{-1}[X], \leq)$, the fibre of X with respect to U ;

(ii) for each $\mathcal{X}$–morphism $f : X \longrightarrow Y$, $A \in TX$ and $B \in TY$,

$Tf(A) \leq B$ iff there exists an $\mathcal{A}$–morphism $a : A \longrightarrow B$ with $U(a) = f$.

A *fibre–functor* of $(\mathcal{A}, U)$ is dually defined.

2.6 Proposition *For a concrete category $(\mathcal{A}, U)$ over $\mathcal{X}$, the object assignment $X \mapsto U^{-1}[X]$ can be extended to a (co–)fibre–functor iff $(\mathcal{A}, U)$ is a (co)fibration .*

Proof We prove the cofibration version :

"$\Rightarrow$" : Consider a U–morphism $f : UA \longrightarrow Y$. Then by assumption there exists an $\mathcal{A}$–morphism $a : A \longrightarrow Tf(A)$ with $U(a) = f$. Now let $g : A \longrightarrow B$ be an $\mathcal{A}$–morphism such that $U(g) = h \cdot f$ for some $h : Y \longrightarrow UB$. Then,
$Th(Tf(A)) = T(h \cdot f)(A) = T(U(g))(A) \leq B$.

"$\Leftarrow$" : If $(\mathcal{A}, U)$ is a cofibration, let $TX = (U^{-1}[X], \leq)$. For any $\mathcal{X}$–morphism $f : X \longrightarrow Y$ and $A \in TX$, define $Tf : TX \longrightarrow TY$ by the assignment $A \mapsto$ the U–final lift of the U–morphism $f : UA \longrightarrow Y$. Then Tf is a Pos–morphism. Let $Tf(A) \leq B$; then there exists a (final) $\mathcal{A}$–morphism $g : A \longrightarrow Tf(A)$, and since $Tf(A) \leq B$, we have an $\mathcal{A}$–morphism $b : Tf(A) \longrightarrow B$ with $U(b) = id_Y$. For the $\mathcal{A}$–morphism $b \cdot g : A \longrightarrow B$, $U(b \cdot g) = f$. Conversely, if there is an $\mathcal{A}$–morphism $a : A \longrightarrow B$ with $U(a) = f$, then since $g : A \longrightarrow Tf(A)$ is final, there is an $\mathcal{A}$–morphism $b : Tf(A) \longrightarrow B$ with $U(b) = id_Y$, i.e. $Tf(A) \leq B$. □

The above Proposition shows that not all concrete categories can be characterised by means of theories. However, we shall show in Chapter 2 that, under certain conditions, small–fibred concrete categories can be fully and concretely embedded into (co)fibrations.

2.7 Proposition *A theory* $T : \mathcal{X} \to \mathbf{Pos}$ *has a dual theory* $\bar{T} : \mathcal{X}^{op} \to \mathbf{Pos}$ *iff* $\mathrm{Mod}(T)$ *is a fibration*.

Proof "$\Rightarrow$" : By the procedure of 1.4, $\bar{T}$ is defined by
$\bar{T}f(\tau) = \bigvee\{\sigma \in TX \mid Tf(\sigma) \leq \tau\}$ for $f : X \longrightarrow Y \in Mor(\mathcal{X})$ and $\tau \in TY$. Then $\bar{T}f(\tau)$ is the initial structure on X with respect to the structured morphism $f : X \longrightarrow (Y, \tau)$: given a map $g : (Z, \delta) \longrightarrow X$ with $T(f \cdot g)(\delta) \leq \tau$, we have $Tf(Tg(\delta)) \leq \tau$, hence $Tg(\delta) \leq \bar{T}f(\tau)$.

"$\Leftarrow$" : If $\mathrm{Mod}(T)$ is a fibration, then for an $\mathcal{X}$–morphism $f : X \longrightarrow Y$ and $\tau \in TY$ the assignment $\tau \mapsto$ the initial lift of $f : X \longrightarrow (Y, \tau)$ defines a theory $\bar{T} : \mathcal{X}^{op} \to \mathbf{Pos}$.

Given $Tf(\sigma) \leq \tau$, for $\sigma \in TX$, we have $\sigma \leq \bar{T}f(Tf(\sigma)) \leq \bar{T}f(\tau)$ (since $\bar{T}f(Tf(\sigma))$ is the initial structure on X with respect to $f : X \longrightarrow (Y, Tf(\sigma))$, and $\bar{T}f$ preserves order). Conversely, if $\sigma \leq \bar{T}f(\tau)$, then $Tf(\sigma) \leq Tf(\bar{T}f(\tau))$, but $Tf(\bar{T}f(\tau)) \leq \tau$ by the definition of $\bar{T}f(\tau)$, hence $Tf(\sigma) \leq \tau$. □

Hence a necessary and sufficient condition for a concrete category $(\mathscr{A}, U)$ to be induced by dual theories is that $(\mathscr{A}, U)$ is a fibration and a cofibration.

2.8 Examples (1) Let **CNorm** denote the category of completely normal spaces and continuous maps. Then, considered as a concrete category over **Set**, **CNorm** is a fibration (see [Csaszar 1978]) .

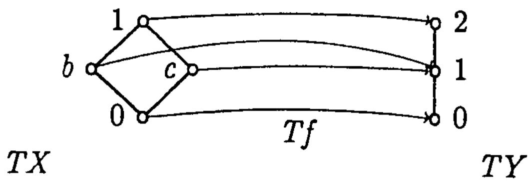
(2) Let **Fsp** denote the category with objects pairs $(X, \mathscr{F})$, where X is a set and $\mathscr{F}$ is a (proper) filter on X , and morphisms the maps $f : (X, \mathscr{F}) \longrightarrow (Y, \mathscr{G})$ where $f^{-1}(\mathscr{G}) \subset \mathscr{F}$. Then **Fsp** is a cofibration, but the fibres are in general not complete lattices.

2.9 Proposition ([Grothendieck 1961], [Wyler 1971b]) *For $(\mathscr{A}, U)$ over $\mathscr{X}$, the following conditions are equivalent :*

(1) $(\mathscr{A}, U)$ is a fibration and a cofibration, and the $\mathscr{A}$-fibres are complete lattices

(2) $(\mathscr{A}, U)$ is a top category, induced by some topological theory $T : \mathscr{X} \to$ **CSLatt** . □

2.10 Example Let $\mathscr{X}$ be a category consisting of two objects X and Y and one non-identity morphism $f : X \longrightarrow Y$. Let TX , TY and Tf be defined as in the following diagram :



The above assignments define a theory $T : \mathscr{X} \to$ **Pos** ; the induced category of T-models is a cofibration and each fibre is a complete lattice. However, Mod(T) is not a fibration, since Tf does not preserve suprema (alternatively, the structured map $f : X \longrightarrow (Y,1)$ has no initial lift).

We end this section by describing BCPos–valued theories (see 2.2) in terms of lift–finally complete categories (cf. Chapter 0 , pp. 2) .

2.11 Proposition *For a concrete category* $(\mathcal{A}, U)$ *the following conditions are equivalent :*

(1) $(\mathcal{A}, U)$ *is a cofibration and lift–finally complete*

(2) $(\mathcal{A}, U)$ *is concretely isomorphic to* Mod(T) *for some theory* $T : \mathcal{X} \to$ BCPos .

Proof (1) ⇒ (2) : Let $(\mathcal{A}, U)$ be lift–finally complete and a cofibration. Then by 2.4 $(\mathcal{A}, U)$ is concretely isomorphic to Mod(T) for some Pos–valued theory T , hence Mod(T) is lift–finally complete. Now suppose that $\{\tau_i\}_{i \in I}$ is a family in TX , for some $X \in Ob(\mathcal{X})$, and for each $i \in I$, $\tau_i \le \sigma$, some $\sigma \in TX$. This means that the structured sink $(id_X : (X, \tau_i) \longrightarrow X)_I$ has a lift, hence it has a final lift $(id_X : (X, \tau_i) \longrightarrow (X, \tau))_I$ in Mod(T) . So, $(X, \tau) = \bigvee_I (X, \tau_i)$ in $(U_T^{-1}[X], \le)$, hence $\tau = \bigvee_I \tau_i$ in TX , i.e. TX is a bounded–complete poset. If $f : X \longrightarrow Y$ is an $\mathcal{X}$–morphism, let $\bigvee_I \tau_i$ be a supremum in TX , i.e. the sink $(id_X : (X, \tau_i) \longrightarrow (X, \bigvee_I \tau_i))_I$ is final in Mod(T) . Now, $(f : (X, \tau_i) \longrightarrow (X, \bigvee_I Tf(\tau_i)))_I$ is a Mod(T)–sink (since $Tf(\bigvee_I \tau_i)$ is an upper bound for all the $Tf(\tau_i)$ and TY is bounded–complete), so by the finality of $(id_X : (X, \tau_i) \longrightarrow (X, \bigvee_I \tau_i))_I$ we have $Tf(\bigvee_I \tau_i) \le \bigvee_I Tf(\tau_i)$, hence $Tf(\bigvee_I \tau_i) = \bigvee_I Tf(\tau_i)$, i.e. Tf preserves suprema.

(2) ⇒ (1) : It is sufficient to show that Mod(T) has the required properties. By 2.4 Mod(T) is a cofibration. Now let $(f_i : (X, \tau_i) \longrightarrow (Y, \sigma))_I$ be a Mod(T)–sink. This

means that for each $i \in I$, $Tf_i(\tau_i) \leq \sigma$, i.e. $\{ Tf_i(\tau_i) \}_I$ is bounded and hence $\bigvee_I Tf_i(\tau_i)$ exists. Now if $g : (Y, \bigvee_I Tf_i(\tau_i)) \longrightarrow (Z, \delta)$ is a map in $\mathcal{X}$ such that $Tg(Tf_i(\tau_i)) \leq \delta$ for each $i \in I$, then $Tg(\bigvee_I Tf_i(\tau_i)) = \bigvee_I Tg(Tf_i(\tau_i))$, since Tg preserves suprema, and $\bigvee_I Tg(Tf_i(\tau_i)) \leq \delta$. Thus g is a T–morphism, i.e. $\bigvee_I Tf_i(\tau_i)$ is the final lift of $(f_i)_I$. □

By duality, $(\mathcal{A}, U)$ is a fibration and lift–initially complete iff $(\mathcal{A}, U)$ is concretely isomorphic to Mod(T) for some $T : \mathcal{X}^{\mathrm{op}} \to$ the category of lowerbound–complete posets and infima–preserving maps.

3. The categories $S(F)$

We describe in theoretic terms those concrete categories which are induced by Set–valued functors. These categories merit special attention since, as mentioned in [Hedrlín, Pultr, Trnková 1966], they contain many every–day concrete categories as full subcategories. Furthermore, they are a fundamental tool in Herrlich's 2–step approach to categorical topology (see [Herrlich 1984]).

3.1 Definition ([Hedrlín, Pultr, Trnková 1966], [Herrlich 1984]) Let $F : \mathcal{X} \to$ **Set** be a functor. Denote by $S(F)$ the concrete category over $\mathcal{X}$ with objects pairs (X, α), where X is an $\mathcal{X}$–object and $\alpha \subset FX$, and morphisms $f : (X, \alpha) \longrightarrow (Y, \beta)$ those $\mathcal{X}$–morphisms for which $Ff(\alpha) \subset \beta$. A concrete category over $\mathcal{X}$ is called *functor–structured* iff it is concretely isomorphic to some category of the form $S(F)$.

3.2 Examples (1) ([Herrlich 1984]) The functor–structured categories over $\odot$ are the complete atomic Boolean algebras.

(2) The category **Rel** of sets with binary relations is functor–structured.

3.3 Theorem ([Menu and Pultr 1974]) *A concrete category* $(\mathscr{A}, U)$ *over* $\mathscr{X}$ *is functor–structured iff the following conditions are satisfied :*

(i) $(\mathscr{A}, U)$ *is topological .*

(ii) Every $\mathscr{A}$ *–fibre is a complete atomic Boolean algebra .*

(iii) If $f : A \longrightarrow B$ *is a final* $\mathscr{A}$ *–morphism, then :*

(a) if B *is discrete, then* A *is discrete*

(b) if A *is an atom, then* B *is also an atom .* □

Now let **CABool** denote the category of complete atomic Boolean algebras and those supremum–preserving maps which in addition reflect bottom elements and preserve atoms (the composition of any two such maps is again such a map).

3.4 Corollary $(\mathscr{A}, U)$ *is functor–structured iff it is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some theory* $T : \mathscr{X} \to \mathbf{CABool}$.

Proof The result follows from 3.3 above and 1.5 of this chapter. □

The above characterisation may be described more explicitly : for a set A , put $PA = (\mathscr{P}A , \subset)$, and for a map $f : A \longrightarrow B$, let $Pf(A)$ be the direct image of A under f . These assignments define a functor $P : \mathbf{Set} \to \mathbf{CSLatt}$ (in fact, P is **CABool**–valued). Now, for a category of form $S(F)$, $F : \mathscr{X} \to \mathbf{Set}$, the composition $P \cdot F$ defines a theory $T : \mathscr{X} \to \mathbf{CSLatt}$ with $\mathrm{Mod}(T) = S(F)$.

4. Concrete functors and transformations

For this section, let S, T be Pos–valued (resp. **CSLatt**–valued) theories on $\mathscr{X}$. Our terminology is motivated by that of [Menu and Pultr 1975].

4.1 Definition A *subtransformation* $\varepsilon : S \longrightarrow T$ is a family of Pos–morphisms $(\varepsilon_X : SX \longrightarrow TX)_{X \in Ob(\mathscr{X})}$ such that for every $\mathscr{X}$–morphism $f : X \longrightarrow Y$ we have $Tf \cdot \varepsilon_X \leq \varepsilon_Y \cdot Sf$.

The following occurs in [Menu and Pultr 1975]. We sketch the proof.

4.2 Proposition

(1) Every subtransformation $\varepsilon : S \longrightarrow T$ *induces a concrete functor* $H_\varepsilon : \mathrm{Mod}(S) \to \mathrm{Mod}(T)$.

(2) Conversely, every concrete functor $H : \mathrm{Mod}(S) \to \mathrm{Mod}(T)$ *induces a subtransformation* $\varepsilon_H : S \longrightarrow T$.

Proof (1) Let $\varepsilon : S \longrightarrow T$ be a subtransformation. The following assignments define a concrete functor $H_\varepsilon : \mathrm{Mod}(S) \to \mathrm{Mod}(T)$:

(a) $H_\varepsilon((X,\tau)) = (X, \varepsilon_X(\tau))$ for $(X,\tau) \in Ob(\mathrm{Mod}(S))$.

(b) If $f : (X,\tau) \longrightarrow (Y,\sigma)$ is an S–morphism, let $H_\varepsilon(f)$ be the morphism $f : (X, \varepsilon_X(\tau)) \longrightarrow (Y, \varepsilon_Y(\sigma))$. (This is a T–morphism : $Sf(\tau) \leq \sigma$, hence, since ε is a subtransformation, $Tf(\varepsilon_X(\tau)) \leq \varepsilon_Y(Sf(\tau)) \leq \varepsilon_Y(\sigma)$.)

(2) Given a concrete functor $H : \mathrm{Mod}(S) \to \mathrm{Mod}(T)$, define, for each $\mathscr{X}$–object X $\varepsilon_X : SX \longrightarrow TX$ by $\varepsilon_X(\tau) = \sigma$, where $(X,\sigma) = H(X,\tau)$. The morphisms ε_X determine a subtransformation $\varepsilon_H : S \longrightarrow T$:

(i) If $id_X : (X,\tau) \longrightarrow (X,\sigma)$ is an S–morphism, then functoriality of H implies that $H(id_X) = id_X : (X, \varepsilon_X(\tau)) \longrightarrow (X, \varepsilon_X(\sigma))$ is a T–morphism, so we have

$\tau \leq \sigma \Rightarrow \varepsilon_X(\tau) \leq \varepsilon_X(\sigma)$.

(ii) For an $\mathscr{X}$-morphism $f : X \longrightarrow Y$, let $\tau \in SX$. Then $f : (X,\tau) \longrightarrow (Y, Sf(\tau))$ is an S-morphism, and hence by functoriality of H , $f : H((X,\tau)) \longrightarrow H((Y, Sf(\tau)))$ is a T-morphism, i.e. from the definitions, $Tf(\varepsilon_X(\tau)) \leq \varepsilon_Y(Sf(\tau))$. Hence $\varepsilon_H : S \longrightarrow T$ is a subtransformation. □

From the constructions in the above proof it follows that subtransformations between theories are in bijective correspondence to concrete functors between their respective categories of models.

4.3 Corollary ([Menu and Pultr 1975]) *For theories S and T, S is naturally isomorphic to T iff* Mod(S) *is concretely isomorphic to* Mod(T) . □

4.4 Proposition *For theories S, $T : \mathscr{X} \to$* CSLatt , *natural transformations $\varepsilon : S \longrightarrow T$ are in bijective correspondence with concrete finality preserving functors H:* Mod(S) $\to$ Mod(T) .

Proof Let $\varepsilon : S \longrightarrow T$ be a natural transformation. In view of 4.2, we must show that $H = H_\varepsilon$ preserves finality : given a final Mod(S)-sink $(f_i : (X_i, \tau_i) \longrightarrow (X, \bigvee_I Sf_i(\tau_i)))_{i \in I}$, we have $\varepsilon_X(\bigvee_I Sf_i(\tau_i)) = \bigvee_I \varepsilon_X(Sf_i(\tau_i))$ (since ε_X preserves suprema) $= \bigvee_I Tf_i(\varepsilon_{X_i}(\tau_i))$ (since ε is a natural transformation) , i.e. $(f_i : (X_i, \varepsilon_{X_i}(\tau_i)) \longrightarrow (X, \varepsilon_X(\bigvee_I Sf_i(\tau_i))))_{i \in I}$ is final in Mod(T) , and coincides with $(H(f_i))_I$ by the definition of H in the proof of 4.2 . Conversely, let $\varepsilon = \varepsilon_H : S \longrightarrow T$ be the induced subtransformation as defined in the proof of 4.2 . First we show that each $\varepsilon_X : SX \longrightarrow TX$ is supremum-preserving : let $\tau = \bigvee_I \tau_i$ be in SX , i.e., the identity-carried sink $(id_X : (X, \tau_i) \longrightarrow (X, \tau))_I$ is final in Mod(S) , hence $(H(id_X) : (X, \varepsilon_X(\tau_i)) \longrightarrow (X, \varepsilon_X(\tau)))_I$ is final in Mod(T) , so $\varepsilon_X(\tau) = \bigvee_I \varepsilon_X(\tau_i)$.

Now consider $f : X \longrightarrow Y$ in $\mathscr{X}$ and let $\tau \in SX$. Since $f : (X,\tau) \longrightarrow (Y, Sf(\tau))$ is final in $\mathrm{Mod}(S)$, it follows that $H(f) = f : (X, \varepsilon_X(\tau)) \longrightarrow (Y, \varepsilon_Y(Sf(\tau)))$ is final in $\mathrm{Mod}(T)$, i.e., $Tf(\varepsilon_X(\tau)) = \varepsilon_Y(Sf(\tau))$. □

Note that 4.1, 4.2, 4.3 and 4.4 above could also have been formulated in terms of dual theories S , $T : \mathscr{X}^{\mathrm{op}} \to \mathbf{CILatt}$. In this case the dual version of 4.4 would have stated a correspondence involving transformations between theories and initiality–preserving concrete functors.

5. (Co)reflective modifications

We characterise (co)reflective modifications of those concrete categories which are determined by theories.

5.1 Definition ([Herrlich 1984]) A full concrete subcategory $(\mathscr{A}, U)$ of $(\mathscr{B}, V)$ is called *concretely (co)reflective* in $(\mathscr{B}, V)$ (or, a *(co)reflective modification* of $(\mathscr{B}, V)$) provided that each $\mathscr{B}$–object B has an identity–carried $\mathscr{A}$–(co)reflection arrow.

Note that if $(\mathscr{A}, U)$ is a (co)reflective modification of $(\mathscr{B}, V)$, then the embedding $E : (\mathscr{A}, U) \hookrightarrow (\mathscr{B}, V)$ has a concrete left adjoint (cf. [Herrlich 1984]).

5.2 It is well–known that a (co)reflective modification of a topological category is again topological. Analagously, every (co)reflective modification of a (co)fibration is a (co)fibration ; in addition, any reflective (resp.coreflective) modification of a concrete category which is a fibration and lift–initially complete (resp. cofibration and lift–finally complete) inherits this property.

Our aim is to give a theoretic interpretation of the following fundamental fact :

5.3 Theorem ([Herrlich 1984]) *If* $(\mathscr{B}, V)$ *is topological, then the following conditions are equivalent for a full concrete subcategory* $(\mathscr{A}, U)$ *of* $(\mathscr{B}, V)$ *:*

(1) $(\mathscr{A}, U)$ *is a (co)reflective modification of* $(\mathscr{B}, V)$

(2) $(\mathscr{A}, U)$ *is initially (finally) closed in* $(\mathscr{B}, V)$. □

The above result does not apply to (co)fibrations, i.e. if $(\mathscr{B}, V)$ is a (co)fibration, then $(\mathscr{A}, U)$ initially (finally) closed in $(\mathscr{B}, V)$ need not imply that $(\mathscr{A}, U)$ is a (co)reflective modification of $(\mathscr{B}, V)$. The result does, however hold for those fibrations (resp. cofibrations) which are lift–initially (resp. lift–finally) complete.

5.4 From a theoretic perspective, 5.3 above shows that a (co)reflective modification of a concrete category determined by some theory is itself determined by a theory which in some sense is a "restriction" of the first theory. Specifically, let $(\mathscr{A}, U)$ be a reflective modification of $\mathrm{Mod}(T)$, for some theory $T : \mathscr{X} \to$ **CSLatt** . Without loss of generality, suppose $(\mathscr{A}, U)$ is a full concrete subcategory of $\mathrm{Mod}(T)$. Let $(X, R(\tau))$ denote the $\mathscr{A}$–reflection of a given $\mathrm{Mod}(T)$–object (X, τ) . Then we have (restating the properties of $(X, R(\tau))$ in theoretic terms) :

(1) $\tau \leq R(\tau)$ in TX ,

(2) for an $\mathscr{A}$–object (Y, σ) and an $\mathscr{X}$–morphism $f : X \longrightarrow Y$, if $Tf(\tau) \leq \sigma$, then $Tf(R(\tau)) \leq \sigma$.

For $X \in Ob(\mathscr{X})$, put $SX = \{ \tau \in TX \mid (X, \tau) \in Ob(\mathscr{A}) \}$, and for an $\mathscr{X}$–morphism $f : X \longrightarrow Y$ define $Sf = Tf|\, SX$. (In fact, $Tf[SX] \subset SY$: let $\tau \in SX$; then $(X, \tau) \in \mathscr{A}$. Since $f : (X, \tau) \longrightarrow (Y, Tf(\tau))$ is final in $\mathrm{Mod}(T)$ and $(\mathscr{A}, U)$ is finally closed in $\mathrm{Mod}(T)$ by 5.3, $(Y, Tf(\tau)) \in Ob(\mathscr{A})$, i.e., $Tf(\tau) \in SY$.) Then $\mathrm{Mod}(S) = (\mathscr{A}, U)$. The above description can be dualized for coreflective modifications.

5.5 Proposition *Given theories* S , $T : \mathscr{X} \to \mathbf{CSLatt}$, $\mathrm{Mod}(S)$ *is a coreflective modification of* $\mathrm{Mod}(T)$ *iff* S *satisfies the following conditions :*

(1) For each $\mathscr{X}$*-object* X , SX *is a supremum-closed subposet of* TX .

(2) For each $\mathscr{X}$*-morphism* $f : X \longrightarrow Y$, $Tf \mid SX = Sf$.

Proof "$\Rightarrow$" : If $\mathrm{Mod}(S)$ is a coreflective modification of $\mathrm{Mod}(T)$, then by **5.3** $\mathrm{Mod}(S)$ is finally closed in $\mathrm{Mod}(T)$, hence condition (1) is satisfied. Condition (2) also follows from **5.3** (see **5.4** above) .

"$\Leftarrow$" : Let (X,τ) be a $\mathrm{Mod}(T)$-object. Put $c(\tau) = \bigvee \{ \alpha \in SX \mid \alpha \leq \tau \}$. Then $c(\tau) \in SX$ by condition (1), and $(X, c(\tau))$ is the $\mathrm{Mod}(S)$-coreflection of (X,τ), since given any $g : (Y,\sigma) \longrightarrow (X,\tau)$ in $\mathrm{Mod}(T)$ with (Y,σ) in $\mathrm{Mod}(S)$, we have $Tg(\sigma) \leq \tau$, and $Tg(\sigma) \in SX$ by (2). Hence $Tg(\sigma) \leq c(\tau)$, i.e., $g : (Y,\sigma) \longrightarrow (X, c(\tau))$ is an S-morphism . □

For theories S , $T : \mathscr{X}^{\mathrm{op}} \to \mathbf{CILatt}$, we obtain an analagous observation, namely, that $\mathrm{Mod}(S)$ is a reflective modification of $\mathrm{Mod}(T)$ iff for each $\mathscr{X}$-object X , SX is an inf-closed subposet of TX , and for each $\mathscr{X}$-morphism $f : X \longrightarrow Y$, $Tf \mid SY = Sf$.

5.6 Remark In view of 4.2, Proposition 5.5 states a correspondence between any (co)reflective modification of a topological category and a subtransformation between theories which is pointwise an infimum (supremum)-closed embedding. An analagous correspondence holds for 5.7 below.

5.7 Proposition *Given theories* S , $T : \mathscr{X} \to \mathbf{BCPos}$, $\mathrm{Mod}(S)$ *is a coreflective modification of* $\mathrm{Mod}(T)$ *iff* S *satisfies the following conditions :*

(1) For each $\mathscr{X}$*-object* X , SX *is a supremum-closed subposet of* TX .

(2) For each $\mathscr{X}$*-morphism* $f : X \longrightarrow Y$, $Tf \mid SX = Sf$.

Proof "$\Rightarrow$" : See proof of 5.5 .

"$\Leftarrow$" : Let (X,τ) be a T–model. Put $\Phi = \{ \sigma \in SX \mid \sigma \leq \tau \}$: this is a bounded subset of TX , hence $\bigvee \Phi$ exists in TX . But then by condition (1) $\bigvee \Phi \in SX$, and $(X, \bigvee \Phi)$ is the $\mathrm{Mod}(S)$–coreflection for (X,τ) , since, given any S–morphism $g:(Y,\delta) \longrightarrow (X,\tau)$, we have $Tg(\delta) \in \Phi$, hence $Tg(\delta) \leq \bigvee \Phi$, i.e. $g:(Y,\delta) \longrightarrow (X, \bigvee \Phi)$ is an S–morphism . □

5.8 Proposition *Let* S , $T : \mathscr{X}^{\mathrm{op}} \to \mathbf{Pos}$ *(resp.* S , $T : \mathscr{X} \to \mathbf{Pos}$ *) be theories. Then* $\mathrm{Mod}(S)$ *is a reflective (resp. coreflective) modification of* $\mathrm{Mod}(T)$ *iff* S *satisfies the following conditions* :

(1) For each $\tau \in TX$, $\bigwedge \{ \sigma \in SX \mid \sigma \geq \tau \}$ *(resp.* $\bigvee \{ \sigma \in SX \mid \sigma \leq \tau \}$ *) exists, and for each* $\mathscr{X}$*–object* X , SX *is infimum–closed (resp. supremum–closed) in* TX .

(2) For each $\mathscr{X}$*–morphism* $f : X \longrightarrow Y$, $Tf \mid SY = Sf$ *(resp.* $Tf \mid SX = Sf$ *)* .

Proof "$\Rightarrow$" : Let $\mathrm{Mod}(S)$ be, say, a coreflective modification of $\mathrm{Mod}(T)$. Given $\tau \in TX$, let $(X, c(\tau))$ denote the $\mathrm{Mod}(T)$–coreflection of $(X, c(\tau))$. So $c(\tau) \in SX$ and $c(\tau) \leq \tau$. Then (by the universal property of $c(\tau)$) we have, for each T–morphism $id_X : (X,\sigma) \longrightarrow (X,\tau)$ with (X,σ) in $\mathrm{Mod}(S)$, $\sigma \leq c(\tau)$, i.e., $c(\tau)$ is an upper bound for the set $\{ \sigma \in SX \mid \sigma \leq \tau \}$; moreover, $c(\tau)$ is the supremum of this set, since $c(\tau) \in \{ \sigma \in SX \mid \sigma \leq \tau \}$. Now, if $\tau = \bigvee_I \sigma_i$ for some family $(\sigma_i)_I$ in SX , then $\tau \leq \bigvee \{ \sigma \in SX \mid \sigma \leq \tau \} = c(\tau) \leq \tau$, i.e., $\tau = c(\tau)$, hence $\tau \in SX$, showing that SX is supremum–closed in TX . Condition (2) follows from the observation that any coreflective modification of a cofibration $(\mathscr{A}, U)$ is finally closed (with respect to singleton sinks) in $(\mathscr{A}, U)$.

"$\Leftarrow$" : Consider (X,τ) in $\mathrm{Mod}(T)$. Since $\bigvee \{ \sigma \in SX \mid \sigma \leq \tau \}$ exists in TX , we may put $c(\tau) = \bigvee \{ \sigma \in SX \mid \sigma \leq \tau \}$. Clearly $c(\tau) \leq \tau$, and $c(\tau) \in SX$ by (1) . The proof of the "$\Leftarrow$" direction in 5.5 (which makes use of condition (2) above) can be applied to show that $c(\tau)$ is indeed the $\mathrm{Mod}(S)$–coreflection for τ . □

CHAPTER 2

MODEL–THEORETIC ASPECTS OF COMPLETIONS

The study of initial and final completions of concrete categories may be considered from two perspectives ([Herrlich 1976, 1979]) : firstly, as a generalisation of the theory of completions of partially–ordered sets (completions of small–fibred concrete categories over the terminal category $\odot$ are the classical completions of partially–ordered sets); secondly, as a means of solving the problem of embedding concrete categories which fail to be topological or convenient into suitable categories possessing these properties.

Since any concrete category determined by a theory is fibre–small, we shall be forced to restrict our attention to fibre–small final completions. Note that even fibre–small concrete categories which have a final completion may fail to have a small–fibred one ([Adámek, Herrlich, Strecker 1979a]). However, those concrete categories which do have small–fibred final completions have been characterised in terms of strengthened fibre–smallness requirements (for definitions, see Chapter 0, pp. 5 – 6) :

Theorem ([Adámek, Herrlich, Strecker 1979a]) *For a concrete category* $(\mathscr{A},U)$ *the following equivalences hold :*

(1) $(\mathscr{A}^*,U^*)$ *is small–fibred iff* $(\mathscr{A},U)$ *is strongly cofibre–small .*

(2) $\mathrm{Univ}(\mathscr{A})$ *is small–fibred iff* $(\mathscr{A},U)$ *is very strongly cofibre–small .*

(3) $\mathrm{Siev}(\mathscr{A})$ *is small–fibred iff* $(\mathscr{A},U)$ *is extremely strongly cofibre–small .* □

1. Theory completions

In this section we describe small–fibred final completions of concrete categories in abstract theoretic terms. Henceforth we shall almost exclusively consider theories $T : \mathscr{X} \to \mathbf{CSLatt}$. This is purely for the sake of convenience, since, where applicable, observations may be reformulated easily in terms of dual theories $T : \mathscr{X}^{\mathrm{op}} \to \mathbf{CILatt}$.

1.1 In order to describe a fibre–small final completion $E : (\mathscr{A}, U) \to (\mathscr{B}, V)$ in theoretic terms, suitable correspondences between the respective $\mathscr{A}$–fibres and $\mathscr{B}$–fibres must be found to characterise the embedding E , i.e., for each $X \in Ob(\mathscr{X})$, $U^{-1}[X]$ and $V^{-1}[X]$ should be related in some manner, and E should be recoverable from this. One possible alternative is to check whether for each $X \in Ob(\mathscr{X})$, $V^{-1}[X]$ is in some sense an order–theoretic completion of $U^{-1}[X]$. In particular, for $(\mathscr{A}, U)$ with a small–fibred final completion, is the Mac Neille (resp. universal final, largest final) completion of $(\mathscr{A}, U)$ (up to concrete isomorphism) the same as the category obtained by taking the order–theoretic Mac Neille (resp. universal final, largest final) completion of each $\mathscr{A}$–fibre ? For $(\mathscr{A}, U)$ over $\odot$, the answer is positive, but in general negative, as shown by the following :

1.2 Examples (1) The category of R_0 (symmetric) spaces is a Mac Neille completion (and a universal initial completion) of T_1–**Top** , the category of T_1–spaces and continuous maps, ([Herrlich and Strecker 1979]), but T_1–fibres are already complete lattices.

(2) ([Adámek, Herrlich, Strecker 1979a]) The largest final completion $\mathscr{L}$ of **Set** (considered as a concrete category over itself) has objects pairs (X, α) , where X is a set and $\alpha \subset \mathscr{P}(X)$ satisfies : $A \in \alpha$ and $B \subset A$ implies $B \in \alpha$. Morphisms $f : (X, \alpha) \longrightarrow (Y, \beta)$ are those maps $f : X \longrightarrow Y$ satisfying : $A \in \alpha$ implies $f[A] \in \beta$. The fibres of $\mathscr{L}$ are obviously very different from the **Set**–fibres.

(3) Any concrete category $(\mathscr{A}, U)$ with an illegitimate Mac Neille completion.

The above examples also show that, in general, for $(\mathscr{A}, U)$ with a small–fibred final completion, a reasonable description of the order–theoretic correspondence between, for example, the $\mathscr{A}$–fibres and the fibres of the corresponding Mac Neille completion of $(\mathscr{A}, U)$, is not feasable. A second approach, which will rely more on the external properties of final completions, is to make use of the correspondence between transformations and concrete functors (see Chapter 1, section 4.) .

1.3 Definition Let $(\mathscr{A}, U)$ be a concrete category. A theory $T : \mathscr{X} \to \mathbf{CSLatt}$ is called a *theory completion* of $(\mathscr{A}, U)$ iff the following conditions are satisfied :

(1) for each $X \in Ob(\mathscr{X})$, $U^{-1}[X]$ is embeddable in TX as a poset;

(2) for each $A, B \in Ob(\mathscr{A})$ and for every $\mathscr{X}$–morphism $f : UA \longrightarrow UB$, $Tf(A) \leq B$ iff there exists an $\mathscr{A}$–morphism $g : A \longrightarrow B$ with $U(g) = f$;

(3) for each $X \in Ob(\mathscr{X})$, the set of all $Tf(A)$ with $f : Y \longrightarrow X$ in $\mathscr{X}$ and $A \in TY$ is supremum–dense in TX .

1.4 Lemma *For a concrete category* $(\mathscr{A}, U)$ *and a theory* $T : \mathscr{X} \to \mathbf{CSLatt}$ *, the following conditions are equivalent :*

(1) There exists a finally dense concrete full embedding $E : (\mathscr{A}, U) \to \mathrm{Mod}(T)$

(2) T *is a theory completion of* $(\mathscr{A}, U)$.

Proof (1) $\Rightarrow$ (2) : Without loss of generality, consider $(\mathscr{A}, U)$ as a full concrete subcategory of $\mathrm{Mod}(T)$. Then it follows immediately that for each $X \in Ob(\mathscr{X})$, $U^{-1}[X] \subset TX$. By concreteness of E the inclusion from $U^{-1}[X]$ to TX preserves order. Fullness of E implies that this inclusion is an embedding. Now let A , $B \in Ob(\mathscr{A})$ and suppose that $f : UA \longrightarrow UB$ is an $\mathscr{X}$–morphism. If $f = U(g)$ for some $\mathscr{A}$–morphism $g : A \longrightarrow B$, then clearly f is a T–morphism, i.e. $Tf(A) \leq B$. The converse follows from the fullness of $\mathscr{A}$ in $\mathrm{Mod}(T)$. Condition (3) of 1.3 above is implied by the final denseness of $(\mathscr{A}, U)$ in $\mathrm{Mod}(T)$.

(2) ⇒ (1) : Without loss of generality, assume that for each $X \in Ob(\mathscr{X})$, $U^{-1}[X]$ is embedded in TX as a subset. Define $E : (\mathscr{A}, U) \to \mathrm{Mod}(T)$ as follows : for $A \in Ob(\mathscr{A})$ let $EA = (UA, A)$; for $f \in Mor(\mathscr{A})$, put $E(f) = U(f)$. In fact, $E(f)$ is a T–morphism by 1.3 (2). Concreteness of E is immediate, and fullness of E follows from 1.3 (2). Since E is full, and U amnestic, E is a full embedding. The final denseness of E follows from 1.3 (3). □

1.5 Proposition *For a concrete category $(\mathscr{A}, U)$ over $\mathscr{X}$ the following conditions are equivalent :*

(1) $(\mathscr{A}, U)$ has a small–fibred Mac Neille completion

(2) $(\mathscr{A}, U)$ has a theory completion .

Proof (1) ⇒ (2) : Trivial.

(2) ⇒ (1) : If $(\mathscr{A}, U)$ has a theory completion T , then by 1.4 above there exists a finally dense concrete full embedding $E : (\mathscr{A}, U) \to \mathrm{Mod}(T)$, hence ([Adámek, Herrlich, Strecker 1979a]) $(\mathscr{A}, U)$ has a small–fibred Mac Neille completion. □

1.6 Definition Let S and T be theory completions of a concrete category $(\mathscr{A}, U)$. Then S is said to be *smaller than* T (alternaltively, T is called an *extension* of S) iff there exists a natural transformation $\eta : T \longrightarrow S$ such that for each $X \in Ob(\mathscr{X})$, the diagram

$$\begin{tikzcd} U^{-1}[X] \arrow[r, hook, "s_X"] \arrow[dr, hook, "t_X"'] & SX \\ & TX \arrow[u, "\eta_X"'] \end{tikzcd}$$

commutes, where s_X and t_X are the respective natural inclusions. As was the case with final completions (cf. Chapter 0, pp. 3 – 4), the smaller–than relation on theory completions is a preorder, and by a *smallest* (resp. *largest*) theory completion of $(\mathscr{A}, U)$ we mean a theory completion which is smaller (resp. larger) than any other completion and such that there is no properly smaller (resp. larger) theory completion.

1.7 Lemma *Let $E_T : (\mathscr{A}, U) \to \mathrm{Mod}(T)$ and $E_S : (\mathscr{A}, U) \to \mathrm{Mod}(S)$ be final completions of $(\mathscr{A}, U)$, and suppose that $H : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$ is a finality preserving concrete functor with $H \cdot E_T = E_S$. Then S is smaller than T.*

Proof By Chapter 1, 4.4 there exists a natural transformation $\varepsilon_H : T \longrightarrow S$. The definition of ε_H together with $E_S = H \cdot E_T$ shows that S is smaller than T. □

1.8 Proposition *If $E^* : (\mathscr{A}, U) \to (\mathscr{A}^*, U^*)$ is a fibre-small Mac Neille completion of $(\mathscr{A}^*, U^*)$, then the theory T^* corresponding to $(\mathscr{A}^*, U^*)$ is the smallest theory completion of $(\mathscr{A}, U)$ with respect to the ordering defined in* 1.6 *above.*

Proof By 1.5 of Chapter 1 $(\mathscr{A}^*, U^*)$ may be written in the form $\mathrm{Mod}(T^*)$, for some $T^* : \mathscr{X} \to$ **CSLatt**. By 1.4 of this section, T^* is a theory completion of $(\mathscr{A}, U)$. Let S be another theory completion of $(\mathscr{A}, U)$. So, by 1.4, there exists a finally dense concrete full embedding $E_S : \mathscr{A} \to \mathrm{Mod}(S)$. Now $\mathrm{Mod}(T^*)$ is (up to concrete isomorphism) the Mac Neille completion of $(\mathscr{A}, U)$, hence there exists a concrete full embedding $G : \mathrm{Mod}(T^*) \to \mathrm{Mod}(S)$ such that $G \cdot E^* = E_S$; equivalently (cf. Chapter 0, pp. 3) there exists a finality preserving concrete functor $H : \mathrm{Mod}(S) \to \mathrm{Mod}(T^*)$ such that $H \cdot E_S = E^*$. Hence by 1.7 above T^* is smaller than S. □

1.9 Proposition *If* $\mathrm{Siev}(\mathscr{A})$ *is fibre-small, then the theory T corresponding to* $\mathrm{Siev}(\mathscr{A})$ *is the largest theory completion of $(\mathscr{A}, U)$.*

Proof Let S be any theory completion of $(\mathscr{A}, U)$. By 1.4, there exists a finally dense concrete full embedding $E_S : (\mathcal{A}, U) \to \mathrm{Mod}(S)$. Since $\mathrm{Mod}(T)$ is the largest final completion of $(\mathcal{A}, U)$, there exists a concrete full embedding $G : \mathrm{Mod}(S) \to \mathrm{Mod}(T)$, hence a finality preserving concrete functor $H : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$ commuting with E_S and $E_T : (\mathcal{A}, U) \to \mathrm{Mod}(T)$. By 1.7, S is smaller than T . □

1.10 Definition For a concrete category $(\mathcal{A}, U)$, a theory completion T of $(\mathcal{A}, U)$ with embeddings $E_X : U^{-1}[X] \longrightarrow TX$ will be called *U–finality preserving* iff the following condition is satisfied : for the U–final lift $(g_i : A_i \longrightarrow A)_I$ of a U–sink $(f_i : UA_i \longrightarrow X)_I$, $E_X(A) = \bigvee_I Tf_i(A_i)$.

The above definition is simply a reformulation in theoretic terms of the condition that the concrete embedding from $(\mathcal{A}, U)$ into $\mathrm{Mod}(T)$ preserves finality. The next observation follows from 1.4 and 1.10 .

1.11 Lemma *For a concrete category $(\mathcal{A}, U)$ the following conditions are equivalent :*

(1) $(\mathcal{A}, U)$ has a U–finality preserving theory completion T

(2) there exists a finality preserving embedding from $(\mathcal{A}, U)$ into $\mathrm{Mod}(T)$. □

1.12 Proposition *If* $\mathrm{Univ}(\mathcal{A})$ *is small–fibred , then the theory T corresponding to* $\mathrm{Univ}(\mathcal{A})$ *is the largest U–finality preserving theory completion of $(\mathcal{A}, U)$* .

Proof As for 1.9, but with the obvious modifications using 1.11 . □

2. Construction of (co)fibration complete categories

2.1 Recall (Chapter 0, pp. 4 – 5) that a Mac Neille completion (if it exists) of a given $(\mathcal{A}, U)$ may be realised as a category of closed sinks (equivalently, closed sources).

Intuitively, a fibration (resp. cofibration) "completion" of $(\mathcal{A},U)$ should embed $(\mathcal{A},U)$ into a full concrete subcategory $(\mathcal{B},V)$ of $(\mathcal{A}^*,U^*)$. Further, the embedding of $(\mathcal{A},U)$ into $(\mathcal{B},V)$ should also be characterizable by an appropriate denseness property.

2.2 Definition A *fibration* (resp. *cofibration*) *completion* of a concrete category $(\mathcal{A},U)$ is a concrete full embedding $E:(\mathcal{A},U)\to(\mathcal{B},V)$, where $(\mathcal{B},V)$ is a fibration (resp. cofibration) and E is dense with respect to V–initial (resp. V–final) morphisms.

2.3 If $E^*:(\mathcal{A},U)\to(\mathcal{A}^*,U^*)$ is the Mac Neille completion of $(\mathcal{A},U)$, then a cofibration completion $(\mathcal{A}_c,U_c)$ of $(\mathcal{A},U)$ may be realised as the following full subcategory of $(\mathcal{A}^*,U^*)$ (the procedure can be dualised to obtain a fibration completion $(\mathcal{A}_f,U_f)$ of $(\mathcal{A},U)$) : the objects of $\mathcal{A}_c$ are those U–sinks (X,S) such that S is the closed hull of some U–morphism $f: UA \longrightarrow X$ (see Chapter 0, pp. 5), and the $\mathcal{A}_c$–morphisms are the sink maps. Note that $(\mathcal{A}_c,U_c)$ is legitimate (since the conglomerate of $\mathcal{A}_c$–objects is codable by the class of U–morphisms). Further, $(\mathcal{A}_c,U_c)$ is a cofibration : if $f: U_c(X,S)\to Y$ is a U_c–morphism, then since S is the closed hull of some U–morphism $g: UA \longrightarrow U_c(X,S)$, the final lift of $f: U_c(X,S)\to Y$ is given by the closed hull of $f\cdot g: UA\longrightarrow Y$. Let $E_c:(\mathcal{A},U)\to(\mathcal{A}_c,U_c)$ be defined as the restriction of E^* on $(\mathcal{A}_c,U_c)$. If (X,S) is an object of $\mathcal{A}_c$, i.e., S is the closed hull of some $f: UA\longrightarrow X$, then $f: E_cA\longrightarrow(X,S)$ is U_c–final, showing that E_c is dense with respect to U_c–final morphisms.

Note that a *fibration completion* of $(\mathcal{A}_c,U_c)$ need not be a cofibration.

2.4 Examples (1) Every partially–ordered set, considered as a concrete category over $\odot$, is both a fibration and a cofibration, hence coincides with its fibration completion and cofibration completion.

(2) Both **Top** and the category of compactly generated spaces are fibration completions of **Comp**, the category of compact spaces.

From the discussion in 2.3 above, every concrete category has a fibration completion and cofibration completion. However, examples of categories which are simultaneously fibration and cofibration completions (as in 2.4 (1) above) seem to be rare, although most concrete categories can be concretely and fully embedded into categories which are fibrations and cofibrations. Those concrete categories with small–fibred fibration (resp. cofibration) completions can be characterised as follows :

2.5 Proposition *For a concrete category* $(\mathscr{A}, U)$ *, the following conditions are equivalent :*

(1) $(\mathscr{A}, U)$ *has a small–fibred cofibration completion*

(2) $(\mathscr{A}, U)$ *has a small–fibred fibration completion*

(3) $(\mathscr{A}, U)$ *is strongly fibre–small (i.e.* $(\mathscr{A}, U)$ *has a fibre–small Mac Neille completion).*

Proof (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) follow from the remarks in 2.3 .

(1) $\Rightarrow$ (3) : We choose to give a proof in theoretic terms : if $E : (\mathscr{A}, U) \to (\mathscr{B}, V)$ is a fibre–small cofibration completion of $(\mathscr{A}, U)$, then (by 2.4 of Chapter 1) $(\mathscr{B}, V)$ is concretely isomorphic to $\mathrm{Mod}(T)$, for some $T : \mathscr{X} \to$ **Pos** . Consider the theory $P : \mathscr{X} \to$ **CSLatt** , defined as follows :

(i) for $X \in Ob(\mathscr{X})$, put $PX = (\mathscr{P}(TX), \subset)$,

(ii) for $f : X \longrightarrow Y$ and $A \subset TX$, let Pf be the image of A under Tf .

For each $X \in Ob(\mathscr{X})$, embed TX into PX by the assignment $\tau \mapsto \{\sigma \in TX \mid \sigma \leq \tau\} = \downarrow \tau$, where $\tau \in TX$. Then the assignments $(X, \tau) \mapsto (X, \downarrow \tau)$ and $f \mapsto f$ define a full concrete embedding $E : \mathrm{Mod}(T) \to \mathrm{Mod}(P)$, since for each $f : X \longrightarrow Y$, $\tau \in TX$ and $\sigma \in TY$, we have $Tf(\tau) \leq \sigma$ iff $Pf(\downarrow \tau) \subset \downarrow \sigma$. Hence $(\mathscr{A}, U)$ is concretely and fully embeddable in $\mathrm{Mod}(P)$, and since $\mathrm{Mod}(P)$ is finally complete, $(\mathscr{A},U)$ has a small-fibred Mac Neille completion.

(2) ⇒ (3) : Dual to (1) ⇒ (3). □

3. Completions of fibrations and cofibrations

In this section we again restrict attention to small-fibred categories. For a given fibration and/or cofibration $(\mathscr{A},U)$, we embed $(\mathscr{A},U)$ into topological categories constructed by a suitable "blowing up" of each $\mathscr{A}$-fibre.

Our first construction will make use of the fact that **CSLatt** is a reflective subcategory of Pos. On posets P, the reflector $R : \mathrm{Pos} \to \mathbf{CSLatt}$ is given by
$RP = (\{ A \subset P \mid A \text{ downward closed} \}, \subset)$.

3.1 Proposition *Let $R : \mathrm{Pos} \to \mathbf{CSLatt}$ be the reflector and consider a theory $T : \mathscr{X} \to \mathrm{Pos}$. Then there exists a natural transformation $\rho : T \longrightarrow RT$ with the following universal property : for any transformation (resp. subtransformation) $\varepsilon : T \longrightarrow S$, where $S : \mathscr{X} \to \mathbf{CSLatt}$, there exists exactly one transformation $\bar{\varepsilon} : RT \longrightarrow S$ (resp. subtransformation $\bar{\varepsilon} : RT \longrightarrow S$ which is pointwise supremum-preserving) making the following diagram commute :*

$$\begin{tikzcd} T \arrow[r, "\varepsilon"] \arrow[d, "\rho"'] & S \\ RT \arrow[ur, dashed, "\bar{\varepsilon}"'] & \end{tikzcd}$$

Proof For each $X \in Ob(\mathscr{X})$, let $\rho_X : TX \longrightarrow RTX$ be the reflection map with respect to TX. If $\varepsilon : T \longrightarrow S$ is a subtransformation, then for each X in $\mathscr{X}$, there exists, by the universal property of ρ_X , a unique $\bar{\varepsilon}_X : RTX \longrightarrow SX$ in **CSLatt** such that $\varepsilon_X = \bar{\varepsilon}_X \cdot \rho_X$. The maps $(\bar{\varepsilon}_X)_{X \in Ob(\mathscr{X})}$ induce a subtransformation : let $f : X \longrightarrow Y$ be in $\mathscr{X}$; we are to verify that $Sf \cdot \bar{\varepsilon}_X \leq \bar{\varepsilon}_X \cdot RTf$. We have

$$\begin{aligned}
Sf \cdot \bar{\varepsilon}_X \cdot \rho_X &= Sf \cdot \varepsilon_X && (\text{since } \varepsilon_X = \bar{\varepsilon}_X \cdot \rho_X\ ) \\
&\leq \varepsilon_Y \cdot Tf && (\text{by the subtransformation property of } \varepsilon\ ) \\
&= \bar{\varepsilon}_Y \cdot \rho_Y \cdot Tf && (\text{since } \varepsilon_Y = \bar{\varepsilon}_Y \cdot \rho_Y\ ) \\
&= \bar{\varepsilon}_Y \cdot RTf \cdot \rho_X && (\text{by the naturality of } \rho\ )\ .
\end{aligned}$$

Now, $Sf \cdot \bar{\varepsilon}_X \cdot \rho_X \leq \bar{\varepsilon}_Y \cdot RTf \cdot \rho_X$ implies that $Sf \cdot \bar{\varepsilon}_X \leq \bar{\varepsilon}_Y \cdot RTf$, since for each X in $\mathscr{X}$, $\rho_X : TX \longrightarrow RTX$ is a supremum–dense embedding. If, on the other hand, ε is a transformation, then, analagous to the above argument, we obtain $Sf \cdot \bar{\varepsilon}_X \cdot \rho_X = \bar{\varepsilon}_Y \cdot RTf \cdot \rho_X$. In this case ρ_X can be cancelled because of its universal property, and so $\bar{\varepsilon}$ is a transformation. In both cases the uniqueness of $\bar{\varepsilon}$ follows from the uniqueness of each ε_X . □

3.2 Corollary *With R and T as in 3.1 above , there exists a concrete full embedding $E : \mathrm{Mod}(T) \to \mathrm{Mod}(RT)$ which has the following universal property : for every theory $S : \mathscr{X} \to$* **CSLatt** *and every concrete functor $F : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$ induced by a natural transformation , there exists exactly one finality preserving concrete functor $\bar{F} : \mathrm{Mod}(RT) \to \mathrm{Mod}(S)$ such that the diagram*

$$\begin{tikzcd}
\mathrm{Mod}(T) \arrow[r, "F"] \arrow[d, "E"'] & \mathrm{Mod}(S) \\
\mathrm{Mod}(RT) \arrow[ur, dashed, "\bar{F}"'] &
\end{tikzcd}$$

commutes .

Proof We show that RT is a theory completion of $\mathrm{Mod}(T)$: since every reflection morphism $\rho_X : TX \longrightarrow RTX$ is a supremum-dense embedding, conditions (1) and (3) of 1.3 in this chapter are satisfied. For 1.3 (2), let $f : X \longrightarrow Y$, $\sigma \in TX$, $\tau \in TY$. Since $(\rho_X)_{X \in Ob(\mathcal{X})}$ is a natural transformation and ρ_Y is an embedding, we have $Tf(\sigma) \leq \tau$ iff $\rho_Y(Tf(\sigma)) = RTf(\rho_X(\sigma)) \leq \rho_Y(\tau)$. So by 1.4 there exists a finally dense concrete full embedding $E : \mathrm{Mod}(T) \to \mathrm{Mod}(RT)$. The universal property of E now follows immediately from 3.1 above and 4.4 of Chapter 1 . □

3.3 Remark The embedding $E : \mathrm{Mod}(T) \to \mathrm{Mod}(RT)$ also has the property that for every theory $S : \mathcal{X} \to \mathbf{CSLatt}$ and every $F : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$ there exists exactly one concrete functor $\bar{F} : \mathrm{Mod}(RT) \to \mathrm{Mod}(S)$ such that the induced subtransformation is pointwise supremum-preserving. This shows that $\mathrm{Mod}(RT)$ is not concretely isomorphic to $\mathrm{Siev}(\mathrm{Mod}(T))$, since if it was, then the subtransformation induced by $F : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$ would be natural.

We now investigate final completions of those cofibrations $(\mathcal{A}, U)$ which are concretely isomorphic to $\mathrm{Mod}(T)$ for theories $T : \mathcal{X} \to \mathbf{Pos}_\vee$, where $\mathbf{Pos}_\vee$ denotes the category of partially-ordered sets and *suprema-preserving* maps. Note that for a poset P (considered as an object of $\mathbf{Pos}_\vee$), the **CSLatt**-reflection of P , denoted by RP , is defined to be the set of all subsets $A \subset P$ satisfying :

(1) $a \in A$ and $b \leq a \Rightarrow b \in A$,

(2) $S \subset A \Rightarrow \bigvee S \in A$.

3.4 Proposition *Let* $R : \mathbf{Pos}_\vee \to \mathbf{CSLatt}$ *be the reflector and consider a theory* $T : \mathcal{X} \to \mathbf{Pos}_\vee$. *Then there exists a natural transformation* $\rho : T \longrightarrow RT$ *with the following property : for every transformation* $\varepsilon : T \longrightarrow S$, $S : \mathcal{X} \to \mathbf{CSLatt}$, *there exists exactly one transformation* $\bar{\varepsilon} : RT \longrightarrow S$ *such that the following diagram commutes :*

$$\begin{tikzcd}
T \arrow[r, "\varepsilon"] \arrow[d, "\rho"'] & S \\
RT \arrow[ur, dashed, "\bar{\varepsilon}"'] &
\end{tikzcd}$$

Proof As in 3.1, for each $X \in Ob(\mathscr{X})$, let $\rho_X : TX \longrightarrow RTX$ be the reflection morphism with respect to TX. Let $\varepsilon : T \longrightarrow S$ be a natural transformation, where $S : \mathscr{X} \to$ **CSLatt**. Since, for each X in $\mathscr{X}$, $\varepsilon_X : TX \longrightarrow SX$ is a morphism in $\mathrm{Pos}_\vee$, i.e., every ε_X preserves suprema, there exists, by the universal property of ρ_X, a unique $\bar{\varepsilon}_X : RTX \longrightarrow SX$ in **CSLatt** such that $\varepsilon_X = \bar{\varepsilon}_X \cdot \rho_X$. The maps $\bar{\varepsilon}_X$ define a natural transformation $\bar{\varepsilon} : RT \longrightarrow S$: analagous to 3.1 we have, for $f : X \longrightarrow Y$ in $\mathscr{X}$,

$$\begin{aligned}
Sf \cdot \bar{\varepsilon}_X \cdot \rho_X &= Sf \cdot \varepsilon_X && (\text{since } \varepsilon_X = \bar{\varepsilon}_X \cdot \rho_X) \\
&= \varepsilon_Y \cdot Tf && (\text{by the naturality of } \varepsilon) \\
&= \bar{\varepsilon}_Y \cdot \rho_Y \cdot Tf && (\text{since } \varepsilon_Y = \bar{\varepsilon}_Y \cdot \rho_Y) \\
&= \bar{\varepsilon}_Y \cdot RTf \cdot \rho_X && (\text{by the naturality of } \rho)\ .
\end{aligned}$$

Now, $Sf \cdot \bar{\varepsilon}_X \cdot \rho_X = \bar{\varepsilon}_Y \cdot RTf \cdot \rho_X$, so $Sf \cdot \bar{\varepsilon}_X = \bar{\varepsilon}_Y \cdot RTf$ by the universal property of ρ_X. The uniqueness of $\bar{\varepsilon}$ follows from the uniqueness of each $\bar{\varepsilon}_X$. □

3.5 Corollary *With* R *and* T *as in* 3.4 *above, there exists a finality preserving concrete full embedding* $E : \mathrm{Mod}(T) \to \mathrm{Mod}(RT)$ *which has the following universal property : for every finality preserving concrete functor* $F : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$ *with* $S : \mathscr{X} \to$ **CSLatt**, *there exists a unique finality preserving concrete functor* $\bar{F} : \mathrm{Mod}(RT) \to \mathrm{Mod}(S)$ *such that the following diagram commutes :*

$$\begin{tikzcd}
\mathrm{Mod}(T) \arrow[r, "F"] \arrow[d, "E"'] & \mathrm{Mod}(S) \\
\mathrm{Mod}(RT) \arrow[ur, dashed, "\bar{F}"'] &
\end{tikzcd}$$

Proof Analagously to 3.2 above, it can be shown that RT is a theory completion of $\mathrm{Mod}(T)$, i.e., the transformation ρ induces a concrete full embedding $E: \mathrm{Mod}(T) \to \mathrm{Mod}(RT)$. In fact, E preserves finality : if a sink $(f_i : (X_i, \tau_i) \longrightarrow (X, \tau))_I$ is final in $\mathrm{Mod}(T)$, then $\tau = \bigvee_I Tf_i(\tau_i)$, hence $\rho_X(\tau) = \rho_X(\bigvee_I Tf_i(\tau_i)) = \bigvee_I \rho_X(Tf_i(\tau_i)) = \bigvee_I RTf_i(\rho_{X_i}(\tau_i))$ by the naturality of ρ and so the sink $(E(f_i))_I = (f_i : (X_i, \rho_{X_i}(\tau_i)) \longrightarrow (X, \rho_X(\tau)))_I$ is final in $\mathrm{Mod}(RT)$. Let $F: \mathrm{Mod}(T) \to \mathrm{Mod}(S)$ be any finality preserving concrete functor with $S: \mathscr{X} \to \mathbf{CSLatt}$. Then, by 4.4 of Chapter 1, since T is $\mathbf{Pos}_{\mathsf{V}}$-valued, F induces a natural transformation $\varepsilon : T \longrightarrow S$. By 3.4 above, there exists a unique transformation $\bar{\varepsilon} : RT \longrightarrow S$ such that $\varepsilon = \bar{\varepsilon} \cdot \rho$, so, by 4.4 of Chapter 1, $\bar{\varepsilon}$ induces a (unique) finality preserving concrete functor $\bar{F} : \mathrm{Mod}(RT) \to \mathrm{Mod}(S)$ such that the above diagram commutes. □

In contrast to Remark 3.3, we obtain, for a fibre-small $\mathrm{Univ}(\mathrm{Mod}(T))$:

3.6 Corollary *With R and T as in 3.4, $E: \mathrm{Mod}(T) \to \mathrm{Mod}(RT)$ coincides with the universal final completion $E: \mathrm{Mod}(T) \to \mathrm{Univ}(\mathrm{Mod}(T))$ iff $\mathrm{Univ}(\mathrm{Mod}(T))$ is small-fibred.*

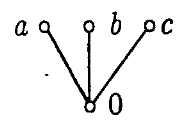
Proof Immediate from 3.4 and 3.5 above. □

3.7 Let $(\mathscr{A}, U)$ be a fibration and a cofibration, and consider the associated $\mathrm{Mod}(T)$, $T: \mathscr{X} \to \mathbf{Pos}$. Note that T has a dual theory $\bar{T} : \mathscr{X}^{\mathrm{op}} \to \mathbf{Pos}$ by 2.7 of Chapter 1. For each X, let $m_X : TX \longrightarrow MX$ be the order-theoretic Mac Neille completion of TX, i.e., MX consists of those $A \subset TX$ for which $l(u(A)) = A$, where $l(A)$ (resp. $u(A)$) is the set of all lower (resp. upper) bounds of A. For $f: X \longrightarrow Y$ and $A \in MX$, define a Pos-morphism $Mf : MX \longrightarrow MY$ by $Mf(A) = l(u(Tf[A]))$, where $Tf[A]$ denotes the image of A under Tf. Note that for $\mathcal{X}$–morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, $M(g \cdot f) = Mg \cdot Mf$: let $A \in MX$; it is easily verified that $M(g \cdot f)(A) \subset Mg(Mf(A))$. For the converse, it is sufficient to show that $Tg[Mf(A)] \subset M(g \cdot f)(A)$. Let $\gamma \in Tg[Mf(A)]$. Then $\gamma = Tg(\sigma)$ for some σ in $l(u(Tf[A]))$. Consider $\delta \in u(Tg[Tf[A]])$, i.e. for all $\alpha \in A$, $\delta \geq Tg(Tf(\alpha))$. Then $\bar{T}g(\delta) \geq Tf(\alpha)$, for all $\alpha \in A$ (by the correspondence between T and $\bar{T}$), hence $\bar{T}g(\delta) \geq \sigma$, and $\gamma = Tg(\sigma) \leq Tg(\bar{T}g(\delta))$. Since T has a dual theory, every Tf preserves suprema, so $Tg(\bar{T}g(\delta)) \leq \delta$ (recall that $\bar{T}g(\delta)$ is defined to be $\bigvee\{\beta \in TX \mid Tf(\beta) \leq \delta\}$). Hence $\gamma \in l(u(Tg[Tf[A]]))$ i.e., $\gamma \in M(g \cdot f)(A)$. It is also easily seen that M preserves identities, and hence is functorial. Moreover, each Mf preserves suprema : given $f : X \longrightarrow Y$, and a family $(A_i)_I$ of elements of MX, since Mf preserves order, we have $\bigvee_I Mf(A_i) \subset Mf(\bigvee_I A_i)$. For the converse, it is sufficient to show that $Tf[\bigvee_I A_i] \subset \bigvee_I Mf(A_i)$. Let $\gamma \in Tf[\bigvee_I A_i]$; then $\gamma = Tf(\sigma)$ for some $\sigma \in l(u(\bigcup_I \mathrm{A}_i)$. If $\delta \in u(\bigcup_I Mf(A_i))$, then it follows that $\delta \geq Tf(\alpha)$ for every $\alpha \in \bigcup_I A_i$, hence $\bar{T}f(\delta) \geq \alpha$ for all $\alpha \in \bigcup_I A_i$, i.e. $\bar{T}f(\delta) \in u(\bigcup_I A_i)$. But then $\sigma \leq \bar{T}f(\delta)$, since $\sigma \in l(u(\bigcup_I A_i)$, so $\gamma = Tf(\sigma) \leq \delta$, i.e. $\gamma \in \bigvee_I Mf(A_i)$. So, M defines a theory $\mathcal{X} \to$ **CSLatt**. Observe also that the maps $m_X : TX \longrightarrow MX$ define a natural transformation $m : T \longrightarrow M$ (where M is considered as a Pos–valued theory) : let $f : X \longrightarrow Y$ be in $\mathcal{X}$. Note that each map m_X sends an element $\sigma \in TX$ to the set $\downarrow \sigma = \{\gamma \in TX \mid \gamma \leq \sigma\}$. From this, and the definition of M, we obtain, for any element $\sigma \in TX$, that $Mf(m_X(\sigma)) = l(u(Tf[\downarrow \sigma])) = \downarrow Tf(\sigma) = m_Y(Tf(\sigma))$. For each X in $\mathcal{X}$, $m_X : TX \longrightarrow MX$ is both infimum–dense and supremum–dense, so the induced $E_M : \mathrm{Mod}(T) \to \mathrm{Mod}(M)$ is both initially and finally dense, hence the Mac Neille completion of $\mathrm{Mod}(T)$.

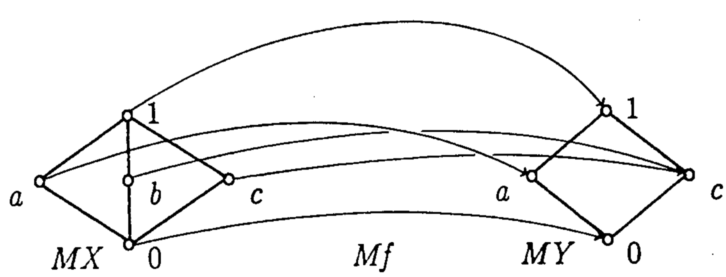
3.8 Proposition *Let* $(\mathcal{A},U)$ *be a fibration and a cofibration. Then there exists a theory* $T^*: \mathcal{X} \to$ **CSLatt** *such that for each* $X \in Ob(\mathcal{X})$, T^*X *is the order-theoretic Mac Neille completion of the fibre* $U^{-1}[X]$ *, and* Mod(T^*) *is the Mac Neille completion of* $(\mathcal{A},U)$.

Proof By 2.7 of Chapter 1, $(\mathcal{A},U)$ is of the form Mod(T) for a theory $T: \mathcal{X} \to$ **Pos** which has a dual theory $\bar{T}: \mathcal{X}^{\mathrm{op}} \to$ **Pos** . Then T^* is given by the theory M constructed in 3.7 above, and T^* satisfies the required conditions, as the considerations in 3.7 show. □

3.9 Examples (1) The category Mod(T) constructed in 2.10 of Chapter 1 is an example of a concrete category which is a cofibration and each fibre is a complete lattice. Since the fibres of Mod(T) are already complete lattices, the procedure described in 3.7 above will yield the category Mod(T) again, but Mod(T) is not finally complete.

(2) Let $\mathcal{X}$ be a category consisting of two objects X and Y and one non-identity morphism $f: X \longrightarrow Y$. Let TX be the poset 

and let $TY = \{a, c, 0\}$, ordered in the same way as TX . Define $Tf: TX \longrightarrow TY$ by $Tf(\sigma) = \sigma$, if $\sigma \neq b$, and let $Tf(b) = c$. Then Tf preserves suprema, i.e., T is even a theory into **BCPos** , and the category Mod(T) induced by T is a cofibration, but not a fibration (the structured morphism $f: X \longrightarrow (Y,c)$ has no initial lift). Now consider the following situation :



In the above diagram, MX and MY are the Mac Neille completions of TX and TY

respectively. Using the definitions in 3.7, the induced map $Mf : MX \longrightarrow MY$ (as shown above) with $Mf \mid TX = Tf$ and $Mf(1) = 1$ does not preserve suprema, alternatively, the structured morphism $f : X \longrightarrow (Y, c)$ has no initial lift.

CHAPTER 3

MODEL–THEORETIC CHARACTERISATIONS OF CONVENIENCE PROPERTIES IN TOPOLOGICAL CATEGORIES

Topological categories are known to possess several pleasant properties; for example, any topological category is complete and cocomplete, wellpowered and cowellpowered ([Herrlich 1982]). However, several important topological categories, such as **Top**, fail to be cartesian closed, and hence are not suitable for investigations in functional analysis, homotopy theory, and topological algebra ([Herrlich and Nel, 1977]).

It is our objective in this chapter to exhibit certain "preservation" properties of a given theory T which are necessary and sufficient for the associated category $\mathrm{Mod}(T)$ to possess a particular "convenience" property. Specifically, we will describe the different types of convenient categories by means of theories $T : \mathscr{X} \to \mathbf{CSLatt}$ sending pullback diagrams into commutative diagrams in **Set** (resp. **CSLatt**) which in addition satisfy some special conditions.

1. (Concrete) cartesian closedness

For the purposes of this section we assume that any given base category $\mathscr{X}$ has finite limits, regular sink factorisations, and is cartesian closed. Our first goal is to characterise, in theoretic terms, the cartesian closed topological categories. The following result is needed :

1.1 Theorem ([Herrlich 1984]) *For a topological category* $(\mathcal{A},U)$ *over* $\mathcal{X}$ *, the following conditions are equivalent :*

(1) $(\mathcal{A},U)$ *is cartesian closed*

(2) for each A *in* $\mathcal{A}$ *, the functor* $A\times -$ *preserves colimits*

(3) regular sinks in $\mathcal{A}$ *are finitely productive*

(4) coproducts and regular epimorphisms in $\mathcal{A}$ *are finitely productive .* □

Condition (3) in the above theorem will be most suitable for our purposes. In order to apply the concept of cartesian closedness to categories of the form $\mathrm{Mod}(T)$, $T:\mathcal{X}\to\mathbf{CSLatt}$, it should be observed that limits and colimits in $\mathrm{Mod}(T)$ are naturally lifted from the base category $\mathcal{X}$, for example, given T–models (X,α) and (Y,β) , the product $(X,\alpha)\times(Y,\beta)$ in $\mathrm{Mod}(T)$ is given by $(X\times Y,\alpha\otimes\beta)$, where $\alpha\otimes\beta$ is the initial structure on $X\times Y$ with respect to the projection source

$$\begin{tikzcd}[row sep=small] & (X,\alpha) \\ X\times Y \arrow[ur, "p_X"] \arrow[dr, "p_Y"'] & \\ & (Y,\beta) \end{tikzcd},$$

i.e., $\alpha\otimes\beta=\bigvee\{\gamma\in T(X\times Y)\mid Tp_X(\gamma)\le\alpha\ ,\ Tp_Y(\gamma)\le\beta\}$. In fact, the operator $\otimes$ induces a family $\otimes_{XY}:TX\times TY\longrightarrow T(X\times Y)$ of maps, where each $\otimes_{XY}$ sends a pair $(\alpha,\beta)\in TX\times TY$ to $\alpha\otimes\beta$. Note also that a sink $(f_i:(X_i,\alpha_i)\longrightarrow(X,\alpha))_I$ in $\mathrm{Mod}(T)$ is regular iff the underlying sink in $\mathcal{X}$ is regular and $\alpha=\bigvee_I Tf_i(\alpha_i)$.

1.2 Examples (1) ([Herrlich 1984]) The cartesian closed topological categories over $\odot$ are the frames, i.e., the complete lattices in which arbitrary suprema distribute over finite infima.

(2) The categories **Lim** (limit spaces) and **Conv** (convergence spaces) are cartesian closed topological ([Herrlich 1982]).

(3) ([Adámek, Herrlich 1985]) **Rel** is a cartesian closed topological category.

In view of 1.2 (1) above, a natural question to ask is whether for an arbitrary cartesian closed topological category $(\mathcal{A},U)$, each $\mathcal{A}$–fibre is a frame (notice that each

Rel–fibre is a frame). The negative answer is given by the following :

1.3 Example The category **Compl** of complemented spaces (a topological space is called *complemented* provided each of its open sets is closed) is a cartesian closed topological subcategory of **Top** ([Herrlich 1983]), but there exist **Compl**–fibres which are not frames : consider the set $\{0,1,2\}$. Let $\tau_1 = \{ \emptyset , \{0\} , \{1,2\} , \{0,1,2\} \}$, $\tau_2 = \{ \emptyset , \{2\} , \{0,1\} , \{0,1,2\} \}$, $\tau_3 = \{ \emptyset , \{1\} , \{2\} , \{0,2\} , \{0,1\} , \{0,1,2\} \}$. Then it can be verified that

$$\begin{tikzcd}
& \{ \emptyset , \{0,1,2\} \} \arrow[dl, dash] \arrow[ddr, dash] & \\
\tau_2 \arrow[dd, dash] & & \\
& & \tau_1 \arrow[ddl, dash] \\
\tau_3 \arrow[dr, dash] & & \\
& \mathcal{P}(\{0,1,2\}) &
\end{tikzcd}$$

is a sublattice of the fibre of $\{0,1,2\}$, showing that this fibre is not even modular.

1.4 Definition Let $\begin{tikzcd}[row sep=small] & X \\ X \times Y \arrow[ur, "p_X"] \arrow[dr, "p_Y"'] & \\ & Y \end{tikzcd}$ be a product in $\mathscr{X}$ and suppose that for each $i \in I$ the diagram

$$\begin{tikzcd}
P_i \arrow[r, "\bar{p}_i"] \arrow[d, "\bar{f}_i"'] & Y_i \arrow[d, "f_i"] \\
X \times Y \arrow[r, "p_Y"'] & Y
\end{tikzcd}$$

is a pullback in $\mathscr{X}$. We say that a theory $T : \mathscr{X} \to$ **CSLatt** sends these pullbacks into a *product covering* family of commutative diagrams

$$\begin{tikzcd} TP_i \arrow[r, "T\bar{p}_i"] \arrow[d, "T\bar{f}_i"'] & TY_i \arrow[d, "Tf_i"] \\ T(X \times Y) \arrow[r, "Tp_Y"'] & TY \end{tikzcd}$$

provided for each $\alpha_i \in TY_i$ $(i \in I)$, and each $\beta \in TX$ there exists some $\gamma_i \in TP_i$ with $T\bar{p}_i(\gamma_i) \leq \alpha_i$ and $\bigvee_I T\bar{f}_i(\gamma_i) = \beta \otimes \bigvee_I Tf_i(\alpha_i)$.

1.5 Theorem *For a topological category* $(\mathscr{A}, U)$ *over* $\mathscr{X}$ *, the following conditions are equivalent :*

(1) $(\mathscr{A}, U)$ *is cartesian closed*

(2) $(\mathscr{A}, U)$ *is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some theory* $T : \mathscr{X} \to \mathbf{CSLatt}$ *sending the pointwise pullback of any regular sink along a projection into a product covering family .*

Proof $(1) \Rightarrow (2)$: Without loss of generality, consider a cartesian closed topological category of the form $\mathrm{Mod}(T)$, $T : \mathscr{X} \to \mathbf{CSLatt}$. By 1.1 above regular sinks in $\mathrm{Mod}(T)$ are finitely productive, equivalently, pointwise pullbacks of regular sinks along projections are regular. Consider the pointwise pullback in $\mathscr{X}$ of a regular sink $(f_i : Y_i \longrightarrow Y)_I$ along a projection $p_Y : X \times Y \longrightarrow Y$ which is of the following form :

$$\begin{tikzcd} X \times Y_i \arrow[r, "\bar{p}_i"] \arrow[d, "id_X \times f_i"'] & Y_i \arrow[d, "f_i"] \\ X \times Y \arrow[r, "p_Y"'] & Y \end{tikzcd}$$

where each $\bar{p}_i : X \times Y_i \longrightarrow Y_i$ is a projection. Let $(\alpha_i)_I$ be a family with $\alpha_i \in TY_i$ for each $i \in I$, and suppose $\beta \in TX$. The sink $(f_i : (Y_i, \alpha_i) \longrightarrow (Y, \bigvee_I Tf_i(\alpha_i)))_I$ is regular in Mod(T), hence by the cartesian closedness of Mod(T), the sink $(id_X \times f_i : (X \times Y_i, \beta \otimes \alpha_i) \longrightarrow (X \times Y, \beta \otimes \bigvee_I Tf_i(\alpha_i)))_I$ is final in Mod(T), i.e., $\beta \otimes \bigvee_I Tf_i(\alpha_i) = \bigvee_I T(id_X \times f_i)(\beta \otimes \alpha_i)$, and clearly $T\bar{p}_i(\beta \otimes \alpha_i) \leq \alpha_i$ for each $i \in I$. Hence the image under T of the above family of diagrams is product covering.

(2) $\Rightarrow$ (1) : It is sufficient to show that regular sinks in Mod(T) are finitely productive, equivalently, that regular sinks in Mod(T) are stable under pullbacks along projections. So, consider a regular sink $(f_i : (Y_i, \alpha_i) \longrightarrow (Y, \alpha))_I$ in Mod(T) and let (X, β) be an arbitrary T-model. Take the pointwise pullback in Mod(T) of $(f_i)_I$ along the projection $p_Y : (X \times Y, \beta \otimes \alpha) \longrightarrow (Y, \alpha)$, which is of the following form :

$$\begin{tikzcd}
(X \times Y_i, \beta \otimes \alpha_i) \arrow[r, "\bar{p}_i"] \arrow[d, "id_X \times f_i"'] & (Y_i, \alpha_i) \arrow[d, "f_i"] \\
(X \times Y, \beta \otimes \alpha) \arrow[r, "p_Y"'] & (Y, \alpha)
\end{tikzcd}$$

Note that since $(f_i : (Y_i, \alpha_i) \longrightarrow (Y, \alpha))_I$ is regular in Mod(T), the underlying sink in $\mathscr{X}$ is regular, and $\alpha = \bigvee_I Tf_i(\alpha_i)$. By the cartesian closedness of $\mathscr{X}$, the sink $(id_Y \times f_i : X \times Y_i \longrightarrow X \times Y)_I$ is regular in $\mathscr{X}$, so it remains to show that $\beta \otimes \bigvee_I Tf_i(\alpha_i) = \bigvee_I T(id_X \times f_i)(\beta \otimes \alpha_i)$. Since T sends the pointwise pullback of the regular sink $(f_i : Y_i \longrightarrow Y)_I$ along p_Y into a product covering family, it follows that for each $i \in I$ there exists $\gamma_i \in T(X \times Y_i)$ such that $T\bar{p}_i(\gamma_i) \leq \alpha_i$ and $\beta \otimes \bigvee_I Tf_i(\alpha_i) = \bigvee_I T(id_X \times f_i)(\gamma_i)$. But, for each $i \in I$, it is clear that $\gamma_i \leq \beta \otimes \alpha_i$, hence, we have

$\bigvee_I T(id_X \times f_i)(\beta \otimes \alpha_i) \leq \beta \otimes \bigvee_I Tf_i(\alpha_i) = \bigvee_I T(id_X \times f_i)(\gamma_i) \leq \bigvee_I T(id_X \times f_i)(\beta \otimes \alpha_i)$, i.e., the sink $(id_X \times f_i : (X \times Y_i, \beta \otimes \alpha_i) \longrightarrow (X \times Y, \beta \otimes \bigvee_I Tf_i(\alpha_i)))_I$ is regular in $\mathrm{Mod}(T)$. □

Recall (Chapter 0, pp. 7) that a concretely cartesian closed topological category is a cartesian closed category which in addition has concrete powers. Concretely cartesian closed topological categories have been characterised as follows :

1.6 Theorem ([Adámek, Herrlich 1985]) *For a topological category* $(\mathscr{A}, U)$ *over* $\mathscr{X}$ *the following conditions are equivalent :*

(1) $(\mathscr{A}, U)$ *is concretely cartesian closed*

(2) final sinks in $\mathscr{A}$ *are finitely productive*

(3) $(\mathscr{A}, U)$ *is cartesian closed and every* $\mathscr{A}$*−morphism with a discrete range has a discrete domain .* □

Note that even for concretely cartesian closed topological categories, fibres need not be frames. The category **Compl** given in 1.3 of this section is a cartesian closed topological c−category, hence has a concretely cartesian closed topological hull (cf. [Herrlich, Strecker 1986]). It can be checked, using τ_1, τ_2 and τ_3 as in 1.3, that the fibre of the set $\{0,1,2\}$ in the concretely cartesian closed topological hull is not a frame.

1.7 Theorem *For a concrete category* $(\mathscr{A}, U)$ *over* $\mathscr{X}$ *, the following conditions are equivalent :*

(1) $(\mathscr{A}, U)$ *is concretely cartesian closed*

(2) $(\mathscr{A}, U)$ *is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some* $T : \mathscr{X} \to \mathbf{CSLatt}$ *sending the pointwise pullback of an arbitrary sink along a projection into a product covering family*

(3) $(\mathscr{A}, U)$ *is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some* $T : \mathscr{X} \to \mathbf{CSLatt}$ *which sends the pointwise pullback of any regular sink along a projection into a product covering family , and sends morphisms to maps which reflect bottom elements .*

Proof (1) $\Leftrightarrow$ (2) : In view of 1.6, we now consider finite products of arbitrary final sinks instead of regular sinks. Hence the proof of 1.5 may be applied, replacing "regular sink in $\mathscr{X}$" by "arbitrary sink in $\mathscr{X}$", and "regular sink in $\mathrm{Mod}(T)$" by "final sink in $\mathrm{Mod}(T)$" .

(1) $\Leftrightarrow$ (3) : Immediate from 1.5 and the equivalence of (1) and (3) in 1.6 . □

The concretely cartesian closed topological categories can also be characterised in terms of the $\otimes$ operator (see pp. 44). In fact, this characterisation may be obtained by an application of *Wyler's taut lift theorem* (cf. [Wyler 1971a]), but a direct proof will be more instructive in the present context.

Let $T : \mathscr{X} \to \mathbf{CSLatt}$ be a theory. Denote by $T(-) \times T(-) : \mathscr{X} \times \mathscr{X} \to \mathbf{CSLatt}$ the composition of the (bi)functors $(T, T) : \mathscr{X} \times \mathscr{X} \to \mathbf{CSLatt} \times \mathbf{CSLatt}$ and $(-) \times (-) : \mathbf{CSLatt} \times \mathbf{CSLatt} \to \mathbf{CSLatt}$, and let $T(-\times-) : \mathscr{X} \times \mathscr{X} \to \mathbf{CSLatt}$ be the composition of $-\times- : \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ and T :

$$\begin{tikzcd}
 & \mathbf{CSLatt} \times \mathbf{CSLatt} \arrow[dr, "-\times-"] & \\
\mathscr{X} \times \mathscr{X} \arrow[ur, "{(T,T)}"] \arrow[rr, shift left, "T(-)\times T(-)"] \arrow[rr, shift right, "T(-\times-)"'] \arrow[dr, "-\times-"'] & & \mathbf{CSLatt} \\
 & \mathscr{X} \arrow[ur, "T"'] &
\end{tikzcd}$$

1.8 Theorem *For a concrete category* $(\mathcal{A},U)$ *over* $\mathcal{X}$ *, the following conditions are equivalent :*

(1) $(\mathcal{A},U)$ *is concretely cartesian closed*

(2) $(\mathcal{A},U)$ *is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some* $T : \mathcal{X} \to \mathbf{CSLatt}$ *such that the maps* $\otimes_{XY} : TX \times TY \longrightarrow T(X \times Y)$ *are the components of a natural transformation* $\otimes : T(-) \times T(-) \longrightarrow T(- \times -)$.

Proof $(1) \Rightarrow (2)$: Consider a concretely cartesian closed topological category of form $\mathrm{Mod}(T)$, $T : \mathcal{X} \to \mathbf{CSLatt}$. By 1.6 above, final sinks in $\mathrm{Mod}(T)$ are finitely productive. Let X , $Y \in Ob(\mathcal{X})$; we show that $\otimes_{XY} : TX \times TY \longrightarrow T(X \times Y)$ preserves arbitrary suprema : let $((\alpha_i, \beta_i))_I$ be a family of elements in $TX \times TY$. By definition, $\bigvee_I (\alpha_i, \beta_i) = (\bigvee_I \alpha_i, \bigvee_I \beta_i)$. The sinks $(id_X : (X, \alpha_i) \longrightarrow (X, \bigvee_I \alpha_i))_I$ and $(id_Y : (X, \beta_i) \longrightarrow (X, \bigvee_I \beta_i))_I$ are respectively final in $\mathrm{Mod}(T)$, hence their product $(id_X \times id_Y : (X \times Y, \alpha_i \otimes \beta_i) \longrightarrow (X \times Y, \bigvee_I \alpha_i \otimes \bigvee_I \beta_i))_I$ is final in $\mathrm{Mod}(T)$, i.e., $\bigvee_I \alpha_i \otimes \bigvee_I \beta_i = \bigvee_I (\alpha_i \otimes \beta_i)$. Now, let $(f,g) : (X,Y) \longrightarrow (W,Z)$ be a morphism in $\mathcal{X} \times \mathcal{X}$. Then the diagram

$$\begin{tikzcd} TX \times TY \arrow[r, "\otimes_{XY}"] \arrow[d, "Tf \times Tg"'] & T(X \times Y) \arrow[d, "T(f \times g)"] \\ TW \times TZ \arrow[r, "\otimes_{WZ}"'] & T(W \times Z) \end{tikzcd}$$

commutes : let $\alpha \in TX$, $\beta \in TY$. So, $f : (X, \alpha) \longrightarrow (W, Tf(\alpha))$ and $g : (Y, \beta) \longrightarrow (Z, Tg(\beta))$ are respectively final in $\mathrm{Mod}(T)$, hence $f \times g : (X \times Y, \alpha \otimes \beta) \longrightarrow (W \times Z, Tf(\alpha) \otimes Tg(\beta))$ is final in $\mathrm{Mod}(T)$, i.e., $T(f \times g)(\alpha \otimes \beta) = Tf(\alpha) \otimes Tg(\beta)$.

$(2) \Rightarrow (1)$: It is sufficient to show that final sinks in $\mathrm{Mod}(T)$ are finitely productive. So, let $(f_i : (X_i, \alpha_i) \longrightarrow (X, \bigvee_I Tf_i(\alpha_i))_I$ be final in $\mathrm{Mod}(T)$ and let (Y, β) be an arbitrary T-model. We show that $(f_i \times id_Y : (X_i \times Y, \alpha_i \otimes \beta) \longrightarrow (X \times Y, \bigvee_I Tf_i(\alpha_i) \otimes \beta))_I$ is final in $\mathrm{Mod}(T)$: in fact, $(\bigvee_I Tf_i(\alpha_i)) \otimes \beta = \bigvee_I (Tf_i(\alpha_i) \otimes \beta)$ (since $\otimes_{XY}$ preserves suprema), and since $\bigvee_I (Tf_i(\alpha_i) \otimes \beta) = \bigvee_I T(f_i \times id_Y)(\alpha_i \otimes \beta)$ (by the naturality of $\otimes$), we have $(\bigvee_I Tf_i(\alpha_i)) \otimes \beta = \bigvee_I T(f_i \times id_Y)(\alpha_i \otimes \beta)$. □

1.9 Remark Observe that the operator $\otimes : T(-) \times T(-) \longrightarrow T(- \times -)$ is an isotransformation iff the associated T preserves finite products. Hence, from 1.8 above it follows that if T preserves finite products, then $\mathrm{Mod}(T)$ is concretely cartesian closed. The converse does not hold : any non-trivial frame is an example of a concretely cartesian closed topological category over $\odot$ for which the associated T does not preserve finite products.

2. Universally topological categories

For the purposes of this section we assume that any given base category $\mathcal{X}$ is finitely complete.

2.1 Definition ([Adámek, Herrlich 1985]) Let $(\mathcal{A}, U)$ be topological over $\mathcal{X}$. Final sinks in $\mathcal{A}$ are said to be *universal* provided for each final sink $(a_i : A_i \longrightarrow A)_I$ in $\mathcal{A}$ and each $\mathcal{A}$-morphism $g : B \longrightarrow A$, the sink $(b_i : B_i \longrightarrow B)_I$ obtained by taking pointwise pullbacks along g

$$\begin{tikzcd} B_i \arrow[r, "\bar{g}"] \arrow[d, "b_i"'] & A_i \arrow[d, "a_i"] \\ B \arrow[r, "g"'] & A \end{tikzcd}$$

is final.

2.2 Definition ([Adámek, Herrlich 1985]) A topological category with universal final sinks is called *universally topological* .

2.3 Examples (1) A concrete category $(\mathscr{A}, U)$ over the terminal category $\odot$ is universally topological iff it is a frame ([Herrlich 1984]).

(2) **Rel**, the category of binary relations, is universally topological.

(3) ([Herrlich 1984]) A category of form $S(F)$ (cf. Chapter 1, section 3.) is universally topological iff F sends pullbacks into weak pullbacks.

2.4 Proposition *Let $(\mathscr{A}, U)$ be a topological category over $\mathscr{X}$. If $(\mathscr{A}, U)$ is universally topological , then each $\mathscr{A}$ –fibre is a frame .*

Proof Given an $\mathscr{X}$–object X , let $B \in U^{-1}[X]$ and suppose $(A_i)_I$ is a family in $U^{-1}[X]$. Recall that $\bigvee_I A_i$ is given by the final lift of the U–sink $(id_X : UA_i \longrightarrow X)_I$, so there exists a final identity–carried $\mathscr{A}$–sink $(a_i : A_i \longrightarrow \bigvee_I A_i)_I$. Note also that $B \wedge \bigvee_I A_i$ is given by the initial structure on X with respect to the U–source

$$\begin{tikzcd} & UB \\ X \arrow[ur, "id_X"] \arrow[dr, "id_X"'] & \\ & U(\bigvee_I A_i) \end{tikzcd}$$

. Since $B \wedge \bigvee_I A_i \leq \bigvee_I A_i$ in $U^{-1}[X]$, there is an $\mathscr{A}$–morphism $b : B \wedge \bigvee_I A_i \longrightarrow \bigvee_I A_i$ such that $U(b) = id_X$. Now, for each $i \in I$, take the pullback of a_i along b :

$$\begin{tikzcd} B \wedge A_i \arrow[r, "\bar{b}_i"] \arrow[d, "\bar{a}_i"'] & A_i \arrow[d, "a_i"] \\ B \wedge \bigvee_I A_i \arrow[r, "b"'] & \bigvee_I A_i \end{tikzcd}$$

Each a_i is identity–carried, and b is also identity–carried, hence all the $\bar{b}_i$ and all the $\bar{a}_i$ are identity–carried, i.e. for each $i \in I'$, $U(\bar{a}_i) = U(\bar{b}_i) = id_X$. Since $(\mathscr{A}, U)$ is topological, the pullback of each a_i along b is given by the initial $\mathscr{A}$–structure on X with respect to the U–source

$$\begin{tikzcd} & U(A_i) \\ X \arrow[ur, "id_X"] \arrow[dr, "id_X"'] & \\ & U(B \wedge \bigvee_I A_i) \end{tikzcd}$$

. Since A_i and $B \wedge \bigvee_I A_i$ are both elements of $U^{-1}[X]$, this initial structure is $(B \wedge \bigvee_I A_i) \wedge A_i = B \wedge A_i$. Now, since $(a_i)_I$ is final, and $(\mathscr{A}, U)$ is universally topological, $(\bar{a}_i : B \wedge A_i \longrightarrow B \wedge \bigvee_I A_i)_I$ is final, i.e., $B \wedge \bigvee_I A_i = \bigvee_I (B \wedge A_i)$. □

By 2.4 above it follows that the fibre–functor associated with any universally topological category is frame–valued, i.e., if $(\mathscr{A}, U)$ is universally topological, then it is concretely isomorphic over $\mathscr{X}$ to Mod(T) for some frame–valued theory T . A natural question to ask is whether every Tf preserves finite infima in a universally topological Mod(T) . The negative answer may be obtained by looking at **Rel** : recall (Chapter 1, 1.3) that **Rel** is concretely isomorphic to Mod(R) , where for each set X $RX = \mathscr{P}(X \times X)$, and given $f : X \longrightarrow Y$ in $\mathscr{X}$, $Rf : RX \longrightarrow RY$ is defined by the assignment $\rho \mapsto (f\times f)[\rho]$, $\rho \in RX$. It is easy to see that in general, for a map $f : X \longrightarrow Y$ and $\rho_1, \rho_2 \in RX$, $(f\times f)[\rho_1 \cap \rho_2] \neq (f\times f)[\rho_1] \cap (f\times f)[\rho_2]$.

We know that universally topological categories are, up to concrete isomorphism, categories of models corresponding to frame-valued theories. Our goal is to determine which such theories characterise these categories. Some additional terminology is required :

2.5 Definition Let $f : L \longrightarrow M$ be a morphism in **CSLatt** . Then

(1) f is said to *preserve downsets* (alternatively, f is called *downset-preserving*) iff for each $a \in L$, $f(\downarrow a) = \downarrow f(a)$ (i.e., for each $a \in L$, $b \in M$, $b \leq f(a) \Rightarrow b = f(c)$ for some $c \in L$ such that $c \leq a$).

(2) f is called *cover-reflecting* iff for each $a \in L$ and each family $(b_i)_I$ in M , $f(a) \leq \bigvee_I b_i \Rightarrow a \leq \bigvee_I c_i$ for some family $(c_i)_I$ in L such that for each $i \in I$, $f(c_i) \leq b_i$.

Note that the cover-reflecting morphisms and the downset-preserving morphisms are both closed under composition. Also, every cover-reflecting morphism reflects bottom (i.e. discrete) elements.

2.6 Definition A commutative diagram

$$\begin{tikzcd} L \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"'] & M \arrow[d, "f"] \\ N \arrow[r, "g"'] & K \end{tikzcd}$$

in **Set** is called a *covering diagram* provided for each $a \in M$ and $b \in N$ with $f(a) = g(b)$ there exists an element $c \in L$ with $\bar{g}(c) = a$ and $\bar{f}(c) = b$. Such an element c is said to *cover* the pair (b,a) .

2.7 Definition A family of diagrams (over a fixed $g : N \longrightarrow K$)

$$\begin{array}{ccc} L_i & \xrightarrow{\bar{g}_i} & M_i \\ {\scriptstyle \bar{f}_i}\downarrow & & \downarrow{\scriptstyle f_i} \\ N & \xrightarrow[g]{} & K \end{array}$$

in **CSLatt** is called *order–covering* provided it satisfies the following condition : for every family $(a_i)_I$ with $a_i \in M_i$ for each $i \in I$, and $b \in N$ with $g(b) \leq \bigvee_I f_i(a_i)$, there exists a family $(c_i)_I$ with $c_i \in L_i$ for each $i \in I$, such that $b = \bigvee_I \bar{f}_i(c_i)$ and $\bar{g}_i(c_i) \leq a_i$ for every $i \in I$.

Denote by **Frm** the full subcategory of **CSLatt** consisting of all frames and suprema–preserving maps. We are now able to give two characterisations of universally topological categories, one in terms of covering diagrams, and the other in terms of order–covering diagrams.

2.8 Theorem *For a concrete category* $(\mathcal{A}, U)$ *over* $\mathcal{X}$ *, the following conditions are equivalent :*

(1) $(\mathcal{A}, U)$ *is universally topological*

(2) $(\mathcal{A}, U)$ *is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some theory* $T : \mathcal{X} \to \mathbf{Frm}$ *sending morphisms into downset–preserving , cover–reflecting maps and pullbacks into covering diagrams*

(3) $(\mathcal{A}, U)$ *is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some theory* $T : \mathcal{X} \to \mathbf{CSLatt}$ *which sends the pointwise pullback of any sink in* $\mathcal{X}$ *into an order–covering family of diagrams .*

Proof $(1) \Rightarrow (2)$: Without loss of generality we may consider a universally topological category of the form $\mathrm{Mod}(T)$ for some $T : \mathcal{X} \to \mathbf{CSLatt}$. By 2.3 above it follows that T is in fact frame–valued. We first show that T sends pullbacks into covering diagrams : consider a pullback

$$\begin{tikzcd} P \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in $\mathcal{X}$. Let $\alpha \in TX$, $\beta \in TY$ such that $Tf(\alpha) = Tg(\beta)$. Let σ denote the initial structure on P with respect to the structured source $P \xrightarrow{\bar{g}} (X, \alpha)$, $P \xrightarrow{\bar{f}} (Y, \beta)$. The diagram

$$\begin{array}{ccc}
(P,\sigma) & \xrightarrow{\bar{g}} & (X,\alpha) \\
{\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle f} \\
(Y,\beta) & \xrightarrow[g]{} & (Z, Tg(\beta) = Tf(\alpha))
\end{array}$$

is a pullback in $\mathrm{Mod}(T)$. Since $f : (X,\alpha) \longrightarrow (Z, Tf(\alpha))$ is final in $\mathrm{Mod}(T)$, and $g : (Y,\beta) \longrightarrow (Z, Tf(\alpha))$ is a T–morphism, it follows from the universal topologicity of $\mathrm{Mod}(T)$ that $\bar{f} : (P,\sigma) \longrightarrow (Y,\beta)$ is final in $\mathrm{Mod}(T)$, i.e., $\beta = T\bar{f}(\sigma)$. By analagous reasoning, it follows that $\bar{g} : (P,\sigma) \longrightarrow (X,\alpha)$ is final in $\mathrm{Mod}(T)$, i.e., $\alpha = T\bar{g}(\sigma)$. So σ covers the pair (β,α) , and the image of the given pullback diagram is a covering diagram. Now, let $f : X \longrightarrow Y$ in $\mathscr{X}$. Then Tf preserves downsets : given $\alpha \in TX$, $\beta \in TY$ such that $\beta \leq Tf(\alpha)$, the diagram

$$\begin{array}{ccc}
(X,\sigma) & \xrightarrow{id_X} & (X,\alpha) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\
(Y,\beta) & \xrightarrow[id_Y]{} & (Y, Tf(\alpha))
\end{array}$$

is a pullback in $\mathrm{Mod}(T)$, where σ denotes the initial structure on X with respect to the structured source $X \begin{array}{l} \xrightarrow{id_X} (X,\alpha) \\ \xrightarrow[f]{} (Y,\beta) \end{array}$, i.e. $\sigma = \bigvee \{ \mu \in TX \mid \mu \leq \alpha , Tf(\mu) \leq \beta \}$. Since $f : (X,\alpha) \longrightarrow (Y, Tf(\alpha))$ is final in $\mathrm{Mod}(T)$, and $id_Y : (Y,\beta) \longrightarrow (Y, Tf(\alpha))$ is a T–morphism, $f : (X,\sigma) \longrightarrow (Y,\beta)$ is final in $\mathrm{Mod}(T)$, i.e., $Tf(\sigma) = \beta$, and $\sigma \leq \alpha$ trivially. It remains to verify that Tf is cover–reflecting : let $\alpha \in TX$ such that $Tf(\alpha) \leq \bigvee_I \beta_i$ for some family $(\beta_i)_I$ in TY . Note that the sink

$(id_Y : (Y, \beta_i) \longrightarrow (Y, \bigvee_I \beta_i))_I$ is final in $\mathrm{Mod}(T)$. Take the pointwise pullback of $(id_Y)_I$ along f :

$$\begin{tikzcd}
(X, \sigma_i) \arrow[r, "f"] \arrow[d, "id_X"'] & (Y, \beta_i) \arrow[d, "id_Y"] \\
(X, \alpha) \arrow[r, "f"'] & (Y, \bigvee_I \beta_i)
\end{tikzcd}$$

For each $i \in I$, $\sigma_i = \bigvee \{ \mu \in TX \mid \mu \leq \alpha , Tf(\mu) \leq \beta_i \}$. Since $(id_Y : (Y, \beta_i) \longrightarrow (Y, \bigvee_I \beta_i))_I$ is final in $\mathrm{Mod}(T)$, and $f : (X, \alpha) \longrightarrow (Y, \bigvee_I \beta_i)$ is a T–morphism, $(id_X : (X, \sigma_i) \longrightarrow (X, \alpha))_I$ is final in $\mathrm{Mod}(T)$, i.e., $\alpha = \bigvee_I \sigma_i$; in addition, for each $i \in I$, $Tf(\sigma_i) \leq \beta_i$, hence Tf is cover–reflecting.

(2) $\Rightarrow$ (1) : It is sufficient to prove that $\mathrm{Mod}(T)$ is universally topological. We begin by showing that pullbacks of final maps in $\mathrm{Mod}(T)$ are final : given a final T–morphism $f : (X, \alpha) \longrightarrow (Z, Tf(\alpha))$ and a T–morphism $g : (Y, \beta) \longrightarrow (Z, Tf(\alpha))$, consider the pullback in $\mathrm{Mod}(T)$ of f along g :

$$\begin{tikzcd}
(P, \sigma) \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"'] & (X, \alpha) \arrow[d, "f"] \\
(Y, \beta) \arrow[r, "g"'] & (Z, Tf(\alpha))
\end{tikzcd}$$

(In the above diagram, $\sigma = \bigvee \{ \mu \in TP \mid T\bar{f}(\mu) \leq \beta , T\bar{g}(\mu) \leq \alpha \}$.) We have $Tg(\beta) \leq Tf(\alpha)$, hence, since Tf preserves downsets, $Tg(\beta) = Tf(\gamma)$ for some $\gamma \in TX$ with $\gamma \leq \alpha$. Since the diagram

$$\begin{array}{ccc} TP & \xrightarrow{T\bar{g}} & TX \\ {\scriptstyle T\bar{f}}\downarrow & & \downarrow{\scriptstyle Tf} \\ TY & \xrightarrow[Tg]{} & TZ \end{array}$$

is a covering diagram, the pair (β,γ) can be covered by some $\delta \in TP$, i.e., $T\bar{g}(\delta) = \gamma$ and $T\bar{f}(\delta) = \beta$ for some $\delta \in TP$. But clearly $\delta \leq \sigma$ since σ is the initial structure on P with respect to $P \begin{array}{l} \xrightarrow{\bar{g}} (X,\alpha) \\ \xrightarrow[\bar{f}]{} (Y,\beta) \end{array}$, hence $\beta = T\bar{f}(\delta) \leq T\bar{f}(\sigma) \leq \beta$, i.e., $\beta = T\bar{f}(\sigma)$ and $\bar{f} : (P,\sigma) \longrightarrow (Y,\beta)$ is final, as required. Now let $(f_i : (X_i,\alpha_i) \longrightarrow (X,\alpha))_I$ be final in Mod(T) (so, $\alpha = \bigvee_I Tf_i(\alpha_i)$) and suppose $g : (Y,\beta) \longrightarrow (X,\alpha)$ is a T-morphism. Taking pointwise pullbacks in Mod(T),

$$\begin{array}{ccc} (Y_i,\sigma_i) & \xrightarrow{\bar{g}_i} & (X_i,\alpha_i) \\ {\scriptstyle \bar{f}_i}\downarrow & & \downarrow{\scriptstyle f_i} \\ (Y,\beta) & \xrightarrow[g]{} & (X,\alpha) \end{array}$$

we must show that $(\bar{f}_i : (Y_i,\sigma_i) \longrightarrow (Y,\beta))_I$ is final, i.e., $\beta = \bigvee_I T\bar{f}_i(\sigma_i)$. Since $T\bar{f}_i(\sigma_i) \leq \beta$ for each $i \in I$, $\bigvee_I T\bar{f}_i(\sigma_i) \leq \beta$. Since Tg is cover-reflecting and $Tg(\beta) \leq \bigvee_I Tf_i(\alpha_i)$ there exists a family $(\beta_i)_I$ in TY such that $\beta \leq \bigvee_I \beta_i$ and for each $i \in I$, $Tg(\beta_i) \leq Tf_i(\alpha_i)$. By the frame law, $\beta = \beta \wedge \bigvee_I \beta_i = \bigvee_I (\beta \wedge \beta_i)$. For each $i \in I$, take the pullback of $f_i : (X_i,\alpha_i) \longrightarrow (X,Tf_i(\alpha_i))$ along $g : (Y,\beta \wedge \beta_i) \longrightarrow (X,Tf_i(\alpha_i))$:

$$\begin{tikzcd}
(Y_i, \gamma_i) \arrow[r, "\bar{g}_i"] \arrow[d, "\bar{f}_i"'] & (X_i, \alpha_i) \arrow[d, "f_i"] \\
(Y, \beta \wedge \beta_i) \arrow[r, "g"'] & (X, Tf_i(\alpha_i))
\end{tikzcd}$$

We have already shown that final morphisms in $\mathrm{Mod}(T)$ are universal, hence for each $i \in I$, $T\bar{f}_i(\gamma_i) = \beta \wedge \beta_i$, and so $\beta = \bigvee_I T\bar{f}_i(\gamma_i)$. But, for each $i \in I$, $\gamma_i \leq \sigma_i$, (since $\gamma_i = \bigvee\{\mu \in TY_i \mid T\bar{f}_i(\mu) \leq \beta \wedge \beta_i, T\bar{g}_i(\mu) \leq \alpha_i\}$, hence $\gamma_i \leq \bigvee\{\mu \in TY_i \mid T\bar{f}_i(\mu) \leq \beta, T\bar{g}_i(\mu) \leq \alpha_i\} = \sigma_i$) and $T\bar{f}_i(\sigma_i) \leq \beta$, so $\beta = \bigvee_I T\bar{f}_i(\gamma_i) \leq \bigvee_I T\bar{f}_i(\sigma_i) \leq \beta$, hence $\beta = \bigvee_I T\bar{f}_i(\sigma_i)$. So, the sink $(\bar{f}_i : (Y_i, \sigma_i) \longrightarrow (Y, \beta))_I$ is final in $\mathrm{Mod}(T)$.

(1) $\Rightarrow$ (3) : We may consider a universally topological category of form $\mathrm{Mod}(T)$, $T : \mathscr{X} \to \mathbf{CSLatt}$. Let $(f_i : X_i \longrightarrow X)_I$ be an arbitrary sink in $\mathscr{X}$ and consider the pointwise pullback along any $g : Y \longrightarrow X$ in $\mathscr{X}$:

$$\begin{tikzcd}
Y_i \arrow[r, "\bar{g}_i"] \arrow[d, "\bar{f}_i"'] & X_i \arrow[d, "f_i"] \\
Y \arrow[r, "g"'] & X
\end{tikzcd}$$

Now, for each $i \in I$, let $\alpha_i \in TX_i$ and let $\beta \in TY$ with $Tg(\beta) \leq \bigvee_I Tf_i(\alpha_i)$. So $(f_i : (X_i, \alpha_i) \longrightarrow (X, \bigvee_I Tf_i(\alpha_i)))_I$ is final in $\mathrm{Mod}(T)$ and $g : (Y, \beta) \longrightarrow (X, \bigvee_I Tf_i(\alpha_i))$ is a T-morphism. Therefore, for the pointwise pullback

$$\begin{tikzcd}
(P_i, \sigma_i) \arrow[r, "\bar{g}_i"] \arrow[d, "\bar{f}_i"'] & (X_i, \alpha_i) \arrow[d, "f_i"] \\
(Y, \beta) \arrow[r, "g"'] & (X, \bigvee_I Tf_i(\alpha_i))
\end{tikzcd}$$

in $\mathrm{Mod}(T)$, the sink $(\bar{f}_i : (P_i, \sigma_i) \longrightarrow (Y, \beta))_I$ is final in $\mathrm{Mod}(T)$ by the universal topologicity of $\mathrm{Mod}(T)$, so $\beta = \bigvee_I T\bar{f}_i(\sigma_i)$ and $T\bar{g}_i(\sigma_i) \leq \alpha_i$ for all $i \in I$, as required.

(3) ⇒ (1) : For this direction it is sufficient to show that final sinks in $\mathrm{Mod}(T)$ are universal. So, let $(f_i : (X_i, \alpha_i) \longrightarrow (X, \bigvee_I Tf_i(\alpha_i)))_I$ be a final sink, and take the pointwise pullback in $\mathrm{Mod}(T)$ of $(f_i)_I$ along an arbitrary T–morphism $g : (Y, \beta) \longrightarrow (X, \bigvee_I Tf_i(\alpha_i))$:

$$\begin{tikzcd}
(P_i, \sigma_i) \arrow[r, "\bar{g}_i"] \arrow[d, "\bar{f}_i"'] & (X_i, \alpha_i) \arrow[d, "f_i"] \\
(Y, \beta) \arrow[r, "g"'] & (X, \bigvee_I Tf_i(\alpha_i))
\end{tikzcd}$$

Note that the family of diagrams

$$\begin{tikzcd}
TP_i \arrow[r, "T\bar{g}_i"] \arrow[d, "T\bar{f}_i"'] & TX_i \arrow[d, "Tf_i"] \\
TY \arrow[r, "Tg"'] & TZ
\end{tikzcd}$$

is order–covering, and since $Tg(\beta) \leq \bigvee_I Tf_i(\alpha_i)$, for every $i \in I$ there exists $\gamma_i \in TP_i$ with $\beta = \bigvee_I T\bar{f}_i(\gamma_i)$ and $T\bar{g}_i(\gamma_i) \leq \alpha_i$. Hence, for each $i \in I$, $T\bar{f}_i(\gamma_i) \leq \beta$, so we have $\gamma_i \leq \sigma_i$ for all $i \in I$ since each $\sigma_i = \bigvee\{\mu \in TP_i \mid T\bar{f}_i(\mu) \leq \beta , T\bar{g}_i(\mu) \leq \alpha_i\}$. Clearly $\bigvee_I T\bar{f}_i(\sigma_i) \leq \beta$, hence, we have $\bigvee_I T\bar{f}_i(\sigma_i) \leq \beta = \bigvee_I T\bar{f}_i(\gamma_i) \leq \bigvee_I T\bar{f}_i(\sigma_i)$, i.e., $\beta = \bigvee_I T\bar{f}_i(\sigma_i)$, and so the sink $(\bar{f}_i : (P_i, \sigma_i) \longrightarrow (Y, \beta))_I$ is final in Mod(T) . □

2.9 Remark In fact, the above theorem extends the well–known result in [Herrlich 1984] which says that a functor–structured category is universally topological iff F sends pullbacks into weak pullbacks, since one easily sees that in **Set**, the covering diagrams are exactly the weak pullbacks.

2.10 Remark We have seen that for a theory T associated with a universally topological category the Tf (for $\mathcal{X}$–morphisms f) do not in general preserve finite infima. It is interesting to see what happens to a universally topological Mod(T) when each Tf preserves finite infima : given $f : X \longrightarrow Y$ in $\mathcal{X}$ such that Tf preserves finite infima, Tf will preserve indiscrete structures (i.e. top elements) and this, together with the fact that Tf preserves downsets, implies that Tf is surjective. Moreover, since Mod(T) is universally topological, it can be shown that the image under T of any $\mathcal{X}$–monomorphism f is injective : given a monomorphism $f : X \longrightarrow Y$ in $\mathcal{X}$ and $\alpha , \beta \in TX$ with $Tf(\alpha) = Tf(\beta)$, the following diagram is a pullback in Mod(T) :

$$\begin{tikzcd}
(X , \alpha \wedge \beta) \arrow[r, "id_X"] \arrow[d, "id_X"'] & (X , \alpha) \arrow[d, "f"] \\
(X , \beta) \arrow[r, "f"'] & (Y , Tf(\alpha) = Tf(\beta))
\end{tikzcd}$$

It follows that $\beta = \alpha \wedge \beta = \alpha$, since $f : (X , \beta) \longrightarrow (Y , Tf(\beta))$ and

$f : (X, \alpha) \longrightarrow (Y, Tf(\alpha))$ are both final T–morphisms. Hence, for each $\mathscr{X}$–monomorphism $f : X \longrightarrow Y$, $Tf : TX \longrightarrow TY$ is an isomorphism, so the structure of those universally topological Mod(T) such that every Tf is finite meet–preserving is rather pathological, for example, for each such T defined on **Set**, $T(\emptyset)$ is isomorphic to every TX .

3. Concrete quasitopoi and topological universes

In this section we assume that any given base category $\mathscr{X}$ is a quasitopos with regular sink factorisations (for definitions, see Chapter 0, pp.6 – 8). We consider those topological categories which are concrete quasitopoi.

Analagous to 2.1 of the previous section, regular sinks in a topological category $(\mathscr{A}, U)$ over $\mathscr{X}$ are said to be *universal* if the pointwise pullback of any regular sink along an arbitrary $\mathscr{A}$–morphism is regular.

3.1 Theorem (Adámek, Herrlich 1985] *For a topological category* $(\mathscr{A}, U)$ *over* $\mathscr{X}$, *the following conditions are equivalent :*

(1) $(\mathscr{A}, U)$ *is a quasitopos*

(2) regular sinks in $\mathscr{A}$ *are universal .* □

3.2 Examples ([Adámek, Herrlich 1985]) (1) **Conv**, the category of convergence spaces, is a quasitopos.

(2) The category **Rere** of reflexive relations is a quasitopos.

3.3 Proposition *If* $(\mathscr{A}, U)$ *is a quasitopos , then every* $\mathscr{A}$ *–fibre is a frame .*

Proof Analagous to the corresponding proof for 2.4 of the previous section, since identity-carried sinks are trivially regular sinks. □

Hence the fibre-functor corresponding to any concrete quasitopos is frame-valued (but does not in general send morphisms to maps which preserve finite infima).

3.4 Theorem *For a topological category* $(\mathcal{A}, U)$ *over* $\mathcal{X}$ *, the following conditions are equivalent :*

(1) $(\mathcal{A}, U)$ *is a quasitopos*

(2) $(\mathcal{A}, U)$ *is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some theory* $T : \mathcal{X} \to \mathbf{CSLatt}$ *which sends the pointwise pullback of any regular sink into an order-covering family of diagrams .*

Proof (1) ⇒ (2) : Without loss of generality, consider a quasitopos of form $\mathrm{Mod}(T)$, $T : \mathcal{X} \to \mathbf{CSLatt}$. Let $(f_i : X_i \longrightarrow X)_I$ be a regular sink in $\mathcal{X}$ and consider the pointwise pullback along any $g : Y \longrightarrow X$ in $\mathcal{X}$:

$$\begin{tikzcd} P_i \arrow[r, "\bar{g}_i"] \arrow[d, "\bar{f}_i"'] & X_i \arrow[d, "f_i"] \\ Y \arrow[r, "g"'] & X \end{tikzcd}$$

Now, for each $i \in I$, let $\alpha_i \in TX_i$ and let $\beta \in TY$ with $Tg(\beta) \leq \bigvee_I Tf_i(\alpha_i)$. So, $(f_i : (X_i, \alpha_i) \longrightarrow (X, \bigvee_I Tf_i(\alpha_i)))_I$ is regular in $\mathrm{Mod}(T)$ and $g : (Y, \beta) \longrightarrow (X, \bigvee_I Tf_i(\alpha_i))$ is a T-morphism. Therefore, for the pointwise pullback

$$\begin{tikzcd}
(P_i, \sigma_i) \arrow[r, "\bar{g}_i"] \arrow[d, "\bar{f}_i"'] & (X_i, \alpha_i) \arrow[d, "f_i"] \\
(Y, \beta) \arrow[r, "g"'] & (X, \bigvee_I Tf_i(\alpha_i))
\end{tikzcd}$$

in $\mathrm{Mod}(T)$, the sink $(\bar{f}_i : (P_i, \sigma_i) \longrightarrow (Y, \beta))_I$ is regular in $\mathrm{Mod}(T)$ by 3.1 above, so $\beta = \bigvee_I T\bar{f}_i(\sigma_i)$ and $T\bar{g}_i(\sigma_i) \leq \alpha_i$ for all $i \in I$, as required.

(2) $\Rightarrow$ (1) : We must show that regular sinks in $\mathrm{Mod}(T)$ are universal. So, let $(f_i : (X_i, \alpha_i) \longrightarrow (X, \bigvee_I Tf_i(\alpha_i)))_I$ be a regular sink, and take the pointwise pullback in $\mathrm{Mod}(T)$ of $(f_i)_I$ along an arbitrary T–morphism $g : (Y, \beta) \longrightarrow (X, \bigvee_I Tf_i(\alpha_i))$:

$$\begin{tikzcd}
(P_i, \sigma_i) \arrow[r, "\bar{g}_i"] \arrow[d, "\bar{f}_i"'] & (X_i, \alpha_i) \arrow[d, "f_i"] \\
(Y, \beta) \arrow[r, "g"'] & (X, \bigvee_I Tf_i(\alpha_i))
\end{tikzcd}$$

The induced sink $(\bar{f}_i : P_i \longrightarrow Y)_I$ is regular in $\mathscr{X}$ (since $\mathscr{X}$ is a quasitopos), so it is sufficient to show that $\beta = \bigvee_I T\bar{f}_i(\sigma_i)$. Note that the family of diagrams

$$\begin{tikzcd}
TP_i \arrow[r, "T\bar{g}_i"] \arrow[d, "T\bar{f}_i"'] & TX_i \arrow[d, "Tf_i"] \\
TY \arrow[r, "Tg"'] & TZ
\end{tikzcd}$$

is order–covering, and since $Tg(\beta) \le \bigvee_I Tf_i(\alpha_i)$, for every $i \in I$ there exists $\gamma_i \in TP_i$ with $\beta = \bigvee_I T\bar{f_i}(\gamma_i)$ and $T\bar{g}_i(\gamma_i) \le \alpha_i$. Hence, for each $i \in I$, $T\bar{f_i}(\gamma_i) \le \beta$, so we have $\gamma_i \le \sigma_i$ for all $i \in I$ since each $\sigma_i = \bigvee \{ \mu \in TP_i \mid T\bar{f_i}(\mu) \le \beta,\ T\bar{g}_i(\mu) \le \alpha_i \}$. Clearly $\bigvee_I T\bar{f_i}(\sigma_i) \le \beta$, hence, we have $\bigvee_I T\bar{f_i}(\sigma_i) \le \beta = \bigvee_I T\bar{f_i}(\gamma_i) \le \bigvee_I T\bar{f_i}(\sigma_i)$, i.e., $\beta = \bigvee_I T\bar{f_i}(\sigma_i)$, and so the sink $(\bar{f_i} : (P_i, \sigma_i) \longrightarrow (Y, \beta)$ is regular in $\mathrm{Mod}(T)$. □

3.5 Remark In [Adámek, Herrlich 1985] the universally topological categories over quasitopoi are characterised as concrete quasitopoi in which every morphism with a discrete range has a discrete domain. Translating this into theoretic terms, we have : a topological category $(\mathscr{A}, U)$ over $\mathscr{X}$ is universally topological iff $(\mathscr{A}, U)$ is concretely isomorphic to $\mathrm{Mod}(T)$ for some $T : \mathscr{X} \to$ **CSLatt** which sends the pointwise pullback of a regular sink into an order–covering family, and which sends morphisms into maps that reflect bottom elements.

Recall (Chapter 0, pp. 8) that a topological category over **Set** is a c–category iff every one–element set has a trivial fibre. In [Herrlich 1984], a *topological universe* is defined to be a topological c–category in which final epi–sinks are universal. In fact, topological universes are precisely those quasitopoi over **Set** which are topological c–categories ([Adámek, Herrlich 1985]). So, one immediately obtains, from 3.4 above and the observation that regular sinks in **Set** are precisely the epi–sinks :

3.6 Theorem *For a topological category* $(\mathscr{A}, U)$ *over* **Set**, *the following conditions are equivalent :*

(1) $(\mathscr{A}, U)$ *is a topological universe*

(2) $(\mathscr{A}, U)$ *is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some* $T : \mathbf{Set} \to$ **CSLatt** *which preserves terminal objects and sends the pointwise pullback of any epi–sink into an order–covering family of diagrams .* □

4. Hereditary topological categories

4.1 Definition ([Herrlich 1988]) A topological c–category $(\mathscr{A}, U)$ over **Set** is called *hereditary* provided coproducts and quotients in $\mathscr{A}$ are hereditary, that is, coproducts and quotients in $\mathscr{A}$ are preserved under pullbacks along embeddings.

4.2 Example The category **PrTop** of pretopological spaces is hereditary ([Herrlich 1988]), but there exist fibres of pretopologies which are not frames (cf. [Schwarz 1979]).

4.3 Proposition ([Herrlich 1988]) *For a topological c–category $(\mathscr{A}, U)$ over* **Set**, *the following conditions are equivalent :*

(1) $(\mathscr{A}, U)$ is hereditary

(2) final epi–sinks in $\mathscr{A}$ are hereditary

(3) final sinks in $\mathscr{A}$ are hereditary . □

4.4 Definition A family of diagrams (over a fixed $g : N \longrightarrow K$)

$$\begin{tikzcd} L_i \arrow[r, "\bar{g}_i"] \arrow[d, "\bar{f}_i"'] & M_i \arrow[d, "f_i"] \\ N \arrow[r, "g"'] & K \end{tikzcd}$$

in **CSLatt** is called *weakly covering* provided it satisfies the following condition : for every family $(a_i)_I$ with $a_i \in M_i$ for each $i \in I$, there exists a family $(c_i)_I$ with $c_i \in L_i$ for each $i \in I$ such that $\bigvee_I T\bar{f}_i(c_i) = \bigvee \{ b \in N \mid g(b) \leq \bigvee_I Tf_i(a_i) \}$ and $\bar{g}_i(c_i) \leq a_i$ for every $i \in I$.

Note that if card $I = 1$ in the above definition, then the associated condition can be reformulated as follows (taking $L = L_1$, $M = M_1$, and $g = g_1$) : for each $a \in M$ there exists $c \in L$ such that $\bar{f}(c) = \bigvee \{ b \in N \mid g(b) \leq f(a) \}$ and $g(c) \leq a$.

4.5 Theorem *For a topological category* $(\mathscr{A}, U)$ *, the following conditions are equivalent :*

(1) $(\mathscr{A}, U)$ *is hereditary*

(2) $(\mathscr{A}, U)$ *is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some theory* $T : \mathbf{Set} \to \mathbf{CSLatt}$ *which preserves terminals , sends pullbacks along embeddings into weakly covering diagrams , and sends embeddings into cover–reflecting morphisms .*

(3) $(\mathscr{A}, U)$ *is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some theory* $T : \mathbf{Set} \to \mathbf{CSLatt}$ *which preserves terminals and sends the pointwise pullback of an arbitrary sink along an embedding into a weakly covering family of diagrams .*

Proof (1) ⇒ (2) : Consider a hereditary topological category of form $\mathrm{Mod}(T)$, $T : \mathbf{Set} \to \mathbf{CSLatt}$. Since $\mathrm{Mod}(T)$ is a topological c–category, T preserves terminal objects. Suppose that the diagram

$$\begin{tikzcd}
P \arrow[r, hook, "\bar{m}"] \arrow[d, "\bar{f}"'] & X \arrow[d, "f"] \\
Y \arrow[r, hook, "m"'] & Z
\end{tikzcd}$$

is a pullback in Set, where $m : Y \hookrightarrow Z$ is an embedding. We are to show that its image under T satisfies the weakly covering property. So, let $\alpha \in TX$, and let β be the initial structure on Y with respect to the structured map $m : Y \hookrightarrow (Z, Tf(\alpha))$, i.e., $\beta = \bigvee \{ \mu \in TY \mid Tm(\mu) \leq Tf(\alpha) \}$. Then, since $f : (X, \alpha) \longrightarrow (Z, Tf(\alpha))$ is final in

Mod(T) , and $m:(Y,\beta)\hookrightarrow(Z,Tf(\alpha))$ is an embedding in Mod(T) , it follows from the heredity of Mod(T) that $\bar{f}:(P,\sigma)\longrightarrow(Y,\beta)$ is final in Mod(T) , where $\sigma=\bigvee\{\,\gamma\in TP\mid T\bar{m}(\gamma)\le\alpha\,,\ T\bar{f}(\gamma)\le\beta\,\}$. Hence $\beta=T\bar{f}(\sigma)$, and so σ is the structure which assures the weak covering property. Now let $m:X\hookrightarrow Y$ be an embedding. Then Tm is cover–reflecting : let $(\alpha_i)_I$ be a family in TY and take $\alpha\in TX$ such that $Tm(\alpha)\le\bigvee_I\beta_i$. Let γ be the initial structure on Y with respect to the structured map $m:X\hookrightarrow(Y,\bigvee_I\beta_i)$. Then $\alpha\le\gamma$, and the following diagram is a pullback in Mod(T) ,

$$\begin{tikzcd}
(X,\sigma_i) \arrow[r, hook, "m"] \arrow[d, "id_X"'] & (Y,\beta_i) \arrow[d, "id_Y"] \\
(X,\gamma) \arrow[r, hook, "m"'] & (Y,\bigvee_I\beta_i)
\end{tikzcd}$$

where for each $i\in I$, $\sigma_i=\bigvee\{\,\mu\in TX\mid Tm(\mu)\le\beta_i\,,\ \mu\le\gamma\,\}$. Since the sink $(id_Y:(Y,\beta_i)\longrightarrow(Y,\bigvee_I\beta_i))_I$ is final in Mod(T) , and $m:(X,\gamma)\hookrightarrow(Y,\bigvee_I\beta_i)$ is an embedding in Mod(T) , the sink $(id_X:(X,\sigma_i)\longrightarrow(X,\gamma))_I$ is final, i.e. $\gamma=\bigvee_I\sigma_i$. Hence $\alpha\le\bigvee_I\sigma_i$, and clearly $Tm(\sigma_i)\le\beta_i$ for each $i\in I$.

(2) $\Rightarrow$ (1) : Since the given theory $T:\mathcal{X}\to\mathbf{CSLatt}$ preserves terminals, it follows that Mod(T) is a topological c–category. To show that Mod(T) is hereditary, we first show that final maps are hereditary. Let $f:(X,\alpha)\longrightarrow(Z,Tf(\alpha))$ be final in Mod(T) and consider the pullback of f along an embedding $m:(Y,\beta)\hookrightarrow(Z,Tf(\alpha))$:

$$\begin{tikzcd}
(P, \sigma) \arrow[r, hook, "\bar{m}"] \arrow[d, "\bar{f}"'] & (Y, \alpha) \arrow[d, "f"] \\
(Y, \beta) \arrow[r, hook, "m"'] & (Z, Tf(\alpha))
\end{tikzcd}$$

where $\sigma = \bigvee \{ \mu \in TP \mid Tm(\mu) \leq \alpha , T\bar{f}(\mu) \leq \beta \}$. Note that $\beta = \bigvee \{ \gamma \in TY \mid Tm(\gamma) \leq Tf(\alpha) \}$. Since the diagram

$$\begin{tikzcd}
TP \arrow[r, hook, "T\bar{m}"] \arrow[d, "T\bar{f}"'] & TX \arrow[d, "Tf"] \\
TY \arrow[r, hook, "Tm"'] & TZ
\end{tikzcd}$$

is weakly covering, there exists $\delta \in TP$ such that $\beta = T\bar{f}(\delta)$ and $T\bar{m}(\delta) \leq \alpha$. But clearly $\delta \leq \sigma$, hence $T\bar{f}(\sigma) = \beta$, i.e., $\bar{f} : (P, \sigma) \longrightarrow (Y, \beta)$ is final in $\mathrm{Mod}(T)$. Now let $(f_i : (X_i, \alpha_i) \longrightarrow (X, \bigvee_I Tf_i(\alpha_i)))_I$ be a final sink in $\mathrm{Mod}(T)$ and take the pointwise pullback in $\mathrm{Mod}(T)$ of $(f_i)_I$ along an embedding $m : (Y, \beta) \hookrightarrow (X, \bigvee_I Tf_i(\alpha_i))$,

$$\begin{tikzcd}
(Y_i, \sigma_i) \arrow[r, hook, "\bar{m}_i"] \arrow[d, "\bar{f}_i"'] & (X_i, \alpha_i) \arrow[d, "f_i"] \\
(Y, \beta) \arrow[r, hook, "m"'] & (X, \bigvee_I Tf_i(\alpha_i))
\end{tikzcd}$$

where each $\sigma_i = \bigvee \{ \mu \in TY_i \mid T\bar{m}_i(\sigma_i) \leq \alpha_i , T\bar{f}_i(\sigma_i) \leq \beta \}$. Since Tm is cover-reflecting, there exists a family $(\beta_i)_I$ in TY such that $\beta = \bigvee_I \beta_i$ and for each

$i \in I$, $Tm(\beta_i) \leq Tf_i(\alpha_i)$. Now, for each $i \in I$, let γ_i denote the initial structure on Y with respect to the structured map $m : Y \hookrightarrow (X, Tf_i(\alpha_i))$. So $\beta_i \leq \gamma_i$ for each $i \in I$, hence $\beta \leq \bigvee_I \gamma_i$, but, since β is initial, and $Tm(\bigvee_I \gamma_i) \leq \bigvee_I Tf_i(\alpha_i)$, it follows that $\beta = \bigvee_I \gamma_i$. Define, for each $i \in I$, δ_i to be the initial structure on Y_i with respect to the structured source $Y_i \begin{smallmatrix} \xrightarrow{\bar{m}_i} (X_i, \alpha_i) \\ \xrightarrow[\bar{f}_i]{} (Y, \gamma_i) \end{smallmatrix}$. So, for each $i \in I$, the diagram

$$\begin{tikzcd}
(Y_i, \delta_i) \arrow[r, hook, "\bar{m}_i"] \arrow[d, "\bar{f}_i"'] & (X_i, \alpha_i) \arrow[d, "f_i"] \\
(Y, \gamma_i) \arrow[r, hook, "m"'] & (X, Tf_i(\alpha_i))
\end{tikzcd}$$

is a pullback in Mod(T) , hence, since we have already shown that final maps are hereditary in Mod(T) , it follows that $\gamma_i = T\bar{f}_i(\delta_i)$ for each $i \in I$. So, $\beta = \bigvee_I \gamma_i = \bigvee_I T\bar{f}_i(\delta_i) \leq \bigvee_I T\bar{f}_i(\sigma_i) \leq \beta$, i.e., the sink $(\bar{f}_i : (Y_i, \sigma_i) \longrightarrow (Y, \beta))_I$ is final in Mod(T) .

(1) $\Rightarrow$ (3) : Consider a hereditary Mod(T) , $T : \textbf{Set} \to \textbf{CSLatt}$. Clearly T preserves terminals. Let $(f_i : X_i \longrightarrow X)_I$ be an arbitrary sink and take its pointwise pullback along any embedding $m : Y \hookrightarrow X$:

$$\begin{tikzcd}
Y_i \arrow[r, hook, "\bar{m}_i"] \arrow[d, "\bar{f}_i"'] & X_i \arrow[d, "f_i"] \\
Y \arrow[r, hook, "m"'] & X
\end{tikzcd}$$

Let $(\alpha_i)_I$ be a family such that for each $i \in I$, $\alpha_i \in TX_i$. Let β be the initial structure on Y with respect to the structured map $m : Y \hookrightarrow (X, \bigvee_I Tf_i(\alpha_i))$. Hence, $m : (Y, \beta) \hookrightarrow (X, \bigvee_I Tf_i(\alpha_i))$ is an embedding in Mod(T) . Since Mod(T) is hereditary, it follows that the sink $(\bar{f}_i : (Y_i, \sigma_i) \longrightarrow (Y, \beta))_I$ is final in Mod(T) , where, for each $i \in I$, $\sigma_i = \bigvee \{ \mu \in TY_i \mid T\bar{m}_i(\mu) \leq \alpha_i , T\bar{f}_i(\mu) \leq \beta \}$. So $\beta = \bigvee_I T\bar{f}_i(\sigma_i)$ and $T\bar{m}_i(\sigma_i) \leq \alpha_i$ for each $i \in I$, which shows that $(\sigma_i)_I$ satisfies the required property.

(3) $\Rightarrow$ (1) : Given a theory T : **Set** $\to$ **CSLatt** of the required form, it is immediate that Mod(T) is a topological c-category. Now, let $(f_i : (X_i, \alpha_i) \longrightarrow (X, \bigvee_I Tf_i(\alpha_i)))_I$ be a final sink in Mod(T) and take its pointwise pullback along an embedding $m : (Y, \beta) \hookrightarrow (X, \bigvee_I Tf_i(\alpha_i))$:

$$
\begin{tikzcd}
(Y_i, \sigma_i) \arrow[r, hook, "\bar{m}_i"] \arrow[d, "\bar{f}_i"'] & (X_i, \alpha_i) \arrow[d, "f_i"] \\
(Y, \beta) \arrow[r, hook, "m"'] & (X, \bigvee_I Tf_i(\alpha_i))
\end{tikzcd}
$$

where, for each $i \in I$, $\sigma_i = \bigvee \{ \mu \in TY_i \mid T\bar{m}_i(\mu) \leq \alpha_i , T\bar{f}_i(\mu) \leq \beta \}$. Since the image of the above family of diagrams under T is a weakly covering family, and β is the initial structure on Y with respect to $m : Y \hookrightarrow (X, \bigvee_I Tf_i(\alpha_i))$, it follows that $\beta = \bigvee_I T\bar{f}_i(\gamma_i)$ for some family $(\gamma_i)_I$ such that for each $i \in I$, $T\bar{m}_i(\gamma_i) \leq \alpha_i$. But it is clear that for each $i \in I$, $\gamma_i \leq \sigma_i$, hence $\beta \leq \bigvee_I T\bar{f}_i(\gamma_i) \leq \bigvee_I T\bar{f}_i(\sigma_i) \leq \beta$, i.e., $\beta = \bigvee_I T\bar{f}_i(\sigma_i)$ and so the sink $(\bar{f}_i : (Y_i, \sigma_i) \longrightarrow (Y, \beta))_I$ is final in Mod(T) . □

Replacing "pullbacks along embeddings" by "pullbacks along monomorphisms", and therefore "weakly covering" by "order-covering", and dropping the preservation of terminals in 4.5 (3), we introduce the following modification of heredity :

4.6 Definition A topological category $(\mathcal{A}, U)$ over **Set** is called *strongly hereditary* provided that final sinks are stable under pullbacks along monomorphisms.

4.7 Example The category **PrTop** of pretopological spaces is hereditary (see 4.2 above), but is not strongly hereditary, as the following Proposition shows.

4.8 Proposition *If $(\mathcal{A}, U)$ is strongly hereditary, then every $\mathcal{A}$-fibre is a frame.*

Proof Analagous to 2.4 of this chapter. □

Since monomorphisms in topological categories, unlike embeddings, need not be initial, it is easy to see that the concept of an order-covering diagram may be applied to characterise the strongly hereditary categories in theoretic terms. In view of the proofs in the previous characterisations, we obtain the following result :

4.9 Theorem *For a concrete category $(\mathcal{A}, U)$ over* **Set**, *the following conditions are equivalent :*

(1) *$(\mathcal{A}, U)$ is strongly hereditary*

(2) *$(\mathcal{A}, U)$ is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some theory* $T : \mathbf{Set} \to \mathbf{Frm}$ *which sends pullbacks along monomorphisms into order-covering diagrams , and sends monomorphisms into cover-reflecting maps*

(3) *$(\mathcal{A}, U)$ is concretely isomorphic to* $\mathrm{Mod}(T)$ *for some theory* $T : \mathbf{Set} \to \mathbf{CSLatt}$ *sending the pointwise pullback of an arbitrary sink along a monomorphism into an order-covering family of diagrams .* □

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